Algebra II Learning Guide
**What Your Student is Learning:** Students will be engaging with identifying and describing inverse and direct variation functions, graphing asymptotes of rational functions, determining whether a rational function has an asymptote and differentiating between vertical, horizontal, and oblique asymptotes.

**Background and Context for Parents:** Two quantities are proportional if they have the same ratio in each instance where they are measured together. Two quantities are inversely proportional if they have the same product in each instance where they are measured together.

A function is a relationship between variables in which each value of the input variable is associated with a unique value of the output variable. Functions can be represented in a variety of ways, such as graphs, tables, equations, or words. Each representation is particularly useful in certain situations. Some important families of functions are developed through transformations of the simplest form of the function.

A single quantity may be represented by many different expressions. The facts about a quantity may be expressed by many different equations or inequalities.

### Graphing Reciprocal Functions

Functions that model inverse variations belong to a family whose parent is the reciprocal function \( f(x) = \frac{1}{x} \). The branches of the parent function are in Quadrants I and III. Reflections are in Quadrants II and IV.

For \( y = \frac{a}{x} \), where \( x \neq 0 \):
- Stretch: \( |a| > 1 \)
- Shrink: \( 0 < |a| < 1 \)
- Reflection in the x-axis: \( a < 0 \)

Reciprocal functions can also be translated horizontally or vertically.

### Asymptotes

An asymptote is a line that the graph approaches as \( x \) or \( y \) increases in absolute value. A vertical asymptote occurs at \( x = a \) if this is a non-removable discontinuity. The graph of a rational function can have any number of vertical asymptotes.

The graph of a rational function can have no more than one **horizontal asymptote**. If the degree of the numerator is \( m \) and the degree of the denominator is \( n \), then:
- If \( m < n \), the graph has a horizontal asymptote, \( y = 0 \).
- If \( m > n \), the graph has no horizontal asymptote.
- If \( m = n \), the graph has a horizontal asymptote, \( y = \frac{a}{b} \), where \( a \) is the numerator’s leading coefficient and \( b \) is the denominator’s leading coefficient.

### Operations With Rational Expressions

**Example:** Find the product of \( \frac{x^2 + 3x + 2}{x + 2} \).

**Solution:** Factor and simplify.

\[
\frac{x^2 + 3x + 2}{x + 2} = \frac{(x - 2)(x + 2)}{(x - 2)} = \frac{x + 2}{x - 2}
\]

The domain of the original expression does not include \(-2\) and \(-3\). Therefore, \(-2\) and \(-3\) are restricted values. Use the Least Common Denominator (LCD) when needed to add or subtract.

**Example:** What is \( \frac{1}{x+3} + \frac{1}{x-3} \)? State any restrictions.

**Solution:** Factor and find LCD.

\[
\frac{1}{x+3} + \frac{1}{x-3} = \frac{1}{x+3} + \frac{1}{x-3} = \frac{x-3 + x+3}{(x+3)(x-3)} = \frac{2x}{x^2 - 9}
\]

The restricted value is \(-3\). 

### Ways to support your student:

- Before giving your student the answer to their question or specific help, ask them “What have you tried so far?, What do you know?, What might be a next step?”
- After your student has solved it, and before you tell them it’s correct or not, have them explain to you how they got their solution and if they think their answer makes sense.

### Online Resources for Students:


- [https://www.khanacademy.org/math/algebra2/x2ec2f6f830c9fb89:rational/x2ec2f6f830c9fb89:rational-graphs/v/asymptotes-of-rational-functions](https://www.khanacademy.org/math/algebra2/x2ec2f6f830c9fb89:rational/x2ec2f6f830c9fb89:rational-graphs/v/asymptotes-of-rational-functions)

Additional GRAPHING PAPER at the end of this packet.
Learning Support for Mathematics

For students that are approaching grade level and have learning gaps/differences in mathematics, provide numerous opportunities for explorations at the concrete (manipulatives) and representational (visual) levels before progressing to the abstract (numbers) level. Students that need learning supports should be provided with:

- Intensive Direct Instruction and daily guided practice
- scaffolded supports
- the use of visuals as models and aids
- numerous opportunities to think out loud
- support to help them understand the why
- use of manipulatives and tools to support understanding
- Bar Modeling Representations to decode word problems
- the use of mnemonics to enhance retention of skills
- daily practice with basic facts
- the presentation of content in varied contexts and varied levels
- opportunities to use diagrams and draw math concepts
- graph paper to support understanding
- numerous opportunities to draw pictures of word problems
- the use of smaller numbers to address number operations
- opportunities for success to build a growth mindset
- computer time to allow for needed practice
- opportunities to engage in metacognition (the building and reinforcing of thinking and reasoning) skills

See examples for each bulleted item on the following pages
· **Intensive Direct Instruction and daily guided practice**
  (Intensive Direct Instruction means to explain the skill / concept to the student with several examples repeatedly to help them understand)

· **Scaffolded Supports**
  (Scaffolded supports means to introduce the skill one step at a time – allowing the student to understand one section part, before moving on to the next part)  ex. 5+ 1=6, 9+1=10, 24+1=25- it is the same as “what number comes after 5, after 9, after 24
  [https://youtu.be/5hWDbSx_kdo](https://youtu.be/5hWDbSx_kdo)

· **Visuals as models and aides**
  (Pictures of objects that can be used to help students understand the math)
  [https://studentsatthecenterhub.org/resource/helping-struggling-students-build-a-growth-min-dset/](https://studentsatthecenterhub.org/resource/helping-struggling-students-build-a-growth-min-dset/)

· **Thinking out loud**
  (Allows students to talk and think about the skills they are learning, which allows them to better remember the skill)
  [https://youtu.be/f-4N7OxSMok](https://youtu.be/f-4N7OxSMok)

· **Understanding the why**
  (When students understand why a strategy works, they will apply it to other skills)  ex. 5x = 5, 45x1= 45, 320x1=320

· **Manipulatives and Tools**
  (Manipulatives can be counters, beans, blocks, etc. – Tools can be rulers, calculators, scales, etc.)  [https://youtu.be/uWBZF-Lyq58](https://youtu.be/uWBZF-Lyq58)

· **Bar Modeling Representations**
  (Bar Modeling Representations consist of visuals that help students understand the skill they are learning. Ex.

<table>
<thead>
<tr>
<th>70</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
</tr>
<tr>
<td>35</td>
</tr>
</tbody>
</table>

[https://youtu.be/TbayTZvS_bc](https://youtu.be/TbayTZvS_bc)
· **Mnemonics**
(Mnemonics consist of strategies to help students remember skills – ex.

![Mnemonic Image](https://example.com/mnemonic_image)

https://youtu.be/dXvvGc9TldY

· **Basic Facts**
(Basic facts include addition, subtraction, division, multiplication facts – ex. 8+2=10, 2+8=10, 10-2=8, 10-8=2 / 2x5=10, 5x2=10, 10/2=5, 10/5=2

https://youtu.be/TbayTZvS_bc

· **Content with varied contexts and varied levels**
Means to show student how to solve a problem different ways to allow them to use the skill that way they understand best

https://youtu.be/FVg9n0l0Gf0

· **Diagrams**
(Diagrams provide students with visuals / pictures that help them solve the problem and they help them read the problem with less words)

https://youtu.be/TbayTZvS_bc

· **Graph paper**
(Graph paper helps students to solve the problem by making it visual / easier to see the answer)

https://youtu.be/mX43cn3IASI

· **Drawing Pictures**
(Drawing pictures allow students to show they can solve the problem without using words that they may not know or be able to write)

https://youtu.be/TbayTZvS_bc
· **Smaller Numbers**
(The use of smaller numbers can help students understand the process of a skill, so that when they move on to bigger numbers, they will see that the process is still the same, they acquire understanding of the skill) ex. 5x = 5, 45x1=45, 320x1=320

· **Growth Mindset**
(A growth mindset is a process that helps to improve intelligence (thinking), ability (skill) and performance (actions). This means that by helping students to develop a growth mindset, we can help them to learn to think and be problem solvers. This is a process that occurs over time by helping them improve by building success over time. [Link](https://studentsatthecenterhub.org/resource/helping-struggling-students-build-a-growth-min dset/)

· **Computer Time**
(Computer time allows students to use websites, games, activities that will help them learn math skills and concepts)
[mathgametime.com](http://www.mathgametime.com), [pbs.com](http://www.pbs.com), [bestkidsolutions.com](http://www.bestkidsolutions.com), [firstinmath.com](http://www.firstinmath.com), [helpingkidsrise.org](http://www.helpingkidsrise.org)

· **Metacognition**
(Metacognition means to help students think about what they are thinking, the steps they are using, the words and numbers that they are using- It helps students to better focus on the skills they are using- it is a process that occurs over time) / [Link](https://youtu.be/HKFOhd5sMEc) / [Link](http://www.spencerauthor.com/metacognition/)
Additional Vocabulary Support
Inverse Variation

Choose the expression from the list that best matches each sentence.

<table>
<thead>
<tr>
<th>combined variation</th>
<th>constant of variation</th>
<th>inverse variation</th>
<th>joint variation</th>
</tr>
</thead>
</table>

1. equations of the form \( xy = k \)

2. when one quantity varies with respect to two or more quantities

3. when one quantity varies directly with two or more quantities

4. the product of two variables in an inverse variation

Choose the expression from the list that best matches each sentence.

<table>
<thead>
<tr>
<th>combined variation</th>
<th>constant of variation</th>
<th>inverse variation</th>
<th>joint variation</th>
</tr>
</thead>
</table>

5. When one quantity increases and the other quantity decreases proportionally, the relationship is an ________________.

6. The function \( z = kxy \) is an example of a ________________.

7. The ________________ is represented by the variable \( k \).

8. Both functions \( z = \frac{kx}{wy} \) and \( z = kxy \) are examples of ________________.

Multiple Choice

9. Which function would be used to model the relationship “\( x \) and \( y \) vary inversely”?

   - A \( y = kx \)
   - B \( z = kxy \)
   - C \( y = \frac{k}{x} \)
   - D \( y = \frac{x}{k} \)

10. Which function would be used to model the relationship “\( z \) varies jointly with \( x \) and \( y \)”?

    - F \( y = kxz \)
    - G \( z = kxy \)
    - H \( y = \frac{k}{z} \)
    - I \( y = \frac{z}{k} \)
8-1 Reteaching
Inverse Variation

The flowchart below shows how to decide whether a relationship between two variables is a direct variation, inverse variation, or neither.

Problem
Do the data in the table represent a direct variation, inverse variation, or neither?

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>5</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>20</td>
<td>10</td>
<td>5</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

As the value of x increases, the value of y decreases, so test the table values in the inverse variation model: \(xy = k\): 1 \cdot 20 = 20, 2 \cdot 10 = 20, 4 \cdot 5 = 20, 5 \cdot 4 = 20. Each product equals the same value, 20, so the data in the table model an inverse variation.

Exercises
Do the data in the table represent a direct variation, inverse variation, or neither?

1. | x  | 5 | 10 | 15 | 20 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>10</td>
<td>20</td>
<td>30</td>
<td>40</td>
</tr>
</tbody>
</table>

2. | x  | 1 | 3 | 4 | 6 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>12</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>
To solve problems involving inverse variation, you need to solve for the constant of variation \( k \) before you can find an answer.

**Problem**

The time \( t \) that is necessary to complete a task varies inversely as the number of people \( p \) working. If it takes 4 h for 12 people to paint the exterior of a house, how long does it take for 3 people to do the same job?

\[
t = \frac{k}{p}
\]

Write an inverse variation. Because time is dependent on people, \( t \) is the dependent variable and \( p \) is the independent variable.

\[
4 = \frac{k}{12}
\]

Substitute 4 for \( t \) and 12 for \( p \).

\[
48 = k
\]

Multiply both sides by 12 to solve for \( k \), the constant of variation.

\[
4 = \frac{48}{p}
\]

Substitute 48 for \( k \). This is the equation of the inverse variation.

\[
t = \frac{48}{3} = 16
\]

Substitute 3 for \( p \). Simplify to solve the equation.

It takes 3 people 16 h to paint the exterior of the house.

**Exercises**

3. The time \( t \) needed to complete a task varies inversely as the number of people \( p \).

It takes 5 h for seven men to install a new roof. How long does it take ten men to complete the job?

4. The time \( t \) needed to drive a certain distance varies inversely as the speed \( r \). It takes 7.5 h at 40 mi/h to drive a certain distance. How long does it take to drive the same distance at 60 mi/h?

5. The cost of each item bought is inversely proportional to the number of items when spending a fixed amount. When 42 items are bought, each costs $1.46. Find the number of items when each costs $2.16.

6. The length \( l \) of a rectangle of a certain area varies inversely as the width \( w \). The length of a rectangle is 9 cm when the width is 6 cm. Determine the length if the width is 8 cm.
8-1  Think About a Plan
Inverse Variation

The spreadsheet shows data that could be modeled by an equation of the form \( PV = k \). Estimate \( P \) when \( V = 62 \).

### Understanding the Problem
1. The data can be modeled by \[ \text{________}. \]

2. What is the problem asking you to determine?

### Planning the Solution
3. What does it mean that the data can be modeled by an inverse variation?

4. How can you estimate the constant of the inverse variation?

5. What is the constant of the inverse variation?

6. Write an equation that you can use to find \( P \) when \( V = 62 \).

### Getting an Answer
7. Solve your equation.

8. What is an estimate for \( P \) when \( V = 62 \)?
Puzzle: Constant of Variation

Inverse Variation

Answer the following questions about inverse and combined variation.

Each ordered pair is from an inverse variation. Find the constant of variation.

1. (2, 1)  
2. (1, 5)  
3. (0.4, 0.5)  
4. (5.2, 0.25)  
5. $\frac{1}{6}$  
6. $\frac{7}{5}$

Suppose that $x$ and $y$ vary inversely. Find the constant of variation.

7. $x = 6$ when $y = \frac{1}{2}$  
8. $x = 3$ when $y = 2$  
9. $x = 0.5$ when $y = 2.2$  
10. $x = 0.2$ when $y = 2$  
11. $x = \frac{2}{3}$ when $y = \frac{2}{3}$  
12. $x = \frac{9}{10}$ when $y = \frac{2}{3}$

Each pair of values is from an inverse variation. Find the missing value.

13. (2, 6), (4, $y$)  
14. (9, 2), ($x$, 3)  
15. (7, 0.2), (5, $y$)  
16. $\frac{2}{3}$, $\frac{5}{6}$

For the following, find $z$ when $x = 2$ and $y = 10$.

17. $z$ varies jointly with $x$ and $y$. When $x = 8$ and $y = 3$, $z = 6$.
18. $z$ varies directly with $x$ and inversely with $y$. When $x = 4$ and $y = 20$, $z = 1$.
19. $z$ varies directly with the square of $y$ and inversely with $x$. When $x = 0.6$ and $y = 0.3$, $z = 0.09$.

The numerical solutions correspond to letters according to the table below.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
<th>J</th>
<th>K</th>
<th>L</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>1.3</td>
<td>1.4</td>
<td>1.5</td>
<td>1.6</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>10</td>
<td>30</td>
<td>50</td>
</tr>
</tbody>
</table>

The numbers below the spaces correspond to the exercise numbers. Write the letter corresponding to the exercise solution in each space. The resulting quotation is by mathematician and philosopher Bertrand Russell.
8-1 Practice
Inverse Variation

Is the relationship between the values in each table a **direct variation**, an **inverse variation**, or **neither**? Write equations to model the direct and inverse variations.

1. \[
\begin{array}{c|c|c|c|c}
\hline
x & 2 & 4 & 5 & 20 \\
\hline
y & 10 & 5 & 4 & 1 \\
\hline
\end{array}
\]

2. \[
\begin{array}{c|c|c|c|c}
\hline
x & 1 & 3 & 7 & 10 \\
\hline
y & 2 & 8 & 20 & 29 \\
\hline
\end{array}
\]

3. \[
\begin{array}{c|c|c|c|c}
\hline
x & 1 & 2 & 5 & 7 \\
\hline
y & 6 & 12 & 30 & 42 \\
\hline
\end{array}
\]

4. \[
\begin{array}{c|c|c|c|c}
\hline
x & 0.2 & 0.5 & 2 & 3 \\
\hline
y & 25 & 62.5 & 250 & 375 \\
\hline
\end{array}
\]

5. \[
\begin{array}{c|c|c|c|c}
\hline
x & \frac{1}{10} & \frac{1}{2} & \frac{3}{2} & 2 \\
\hline
y & 31 & 7 & 3 & \frac{5}{2} \\
\hline
\end{array}
\]

6. \[
\begin{array}{c|c|c|c|c}
\hline
x & 3 & 1.5 & 0.5 & 0.3 \\
\hline
y & 5 & 10 & 30 & 50 \\
\hline
\end{array}
\]

Suppose that \(x\) and \(y\) vary inversely. Write a function that models each inverse variation. Graph the function and find \(y\) when \(x = 10\).

7. \(x = 7\) when \(y = 2\)  
8. \(x = 4\) when \(y = 0.2\)  
9. \(x = \frac{1}{3}\) when \(y = \frac{9}{10}\)

10. The students in a school club decide to raise money by selling hats with the school mascot on them. The table below shows how many hats they can expect to sell based on how much they charge per hat in dollars.

<table>
<thead>
<tr>
<th>Price per Hat ((p))</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hats Sold ((h))</td>
<td>72</td>
<td>60</td>
<td>45</td>
<td>40</td>
</tr>
</tbody>
</table>

**a.** What is a function that models the data? 
**b.** How many hats can the students expect to sell if they charge $7.50 per hat?

11. The minimum number of carpet rolls \(n\) needed to carpet a house varies directly as the house's square footage \(h\) and inversely with the square footage \(r\) in one roll. It takes a minimum of two 1200-ft\(^2\) carpet rolls to cover 2300 ft\(^2\) of floor. What is the minimum number of 1200-ft\(^2\) carpet rolls you would need to cover 2500 ft\(^2\) of floor? Round your answer up to the nearest half roll.
8-1 Practice
Inverse Variation

Is the relationship between the values in each table a direct variation, an inverse variation, or neither? Write an equation to model the direct and inverse variations.

1. \[
\begin{array}{|c|c|}
\hline
x & y \\
\hline
0.1 & 3 \\
3 & 0.1 \\
6 & 0.05 \\
24 & 0.0125 \\
\hline
\end{array}
\]

2. \[
\begin{array}{|c|c|}
\hline
x & y \\
\hline
1 & 3 \\
2 & 6 \\
5 & 15 \\
6 & 18 \\
\hline
\end{array}
\]

3. \[
\begin{array}{|c|c|}
\hline
x & y \\
\hline
0 & 1 \\
2 & 5 \\
4 & 7 \\
6 & 8 \\
\hline
\end{array}
\]

Suppose that \(x\) and \(y\) vary inversely. Write a function that models each inverse variation. Graph the function and find \(y\) when \(x = 10\).

4. \(x = 2\) when \(y = -4\)
5. \(x = -9\) when \(y = -1\)
6. \(x = 1.5\) when \(y = 10\)

7. Suppose the table at the right shows the time \(t\) it takes to drive home when you travel at various average speeds \(s\).
   a. Write a function that models the relationship between the speed and the time it takes to drive home.
   b. At what speed would you need to drive to get home in 50 min or \(\frac{5}{6}\) h?

<table>
<thead>
<tr>
<th>Time (t) (h)</th>
<th>Speed (s) (mi/h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{1}{6})</td>
<td>60</td>
</tr>
<tr>
<td>(\frac{1}{4})</td>
<td>40</td>
</tr>
<tr>
<td>(\frac{1}{3})</td>
<td>30</td>
</tr>
<tr>
<td>(\frac{3}{4})</td>
<td>13.3</td>
</tr>
</tbody>
</table>
Use combined variation to solve each problem.

8. The height $h$ of a cylinder varies directly with the volume of the cylinder and inversely with the square of the cylinder's radius, $r$, with the constant equal to $\frac{1}{p}$.
   a. Write a formula that models this combined variation.
   b. What is the height of a cylinder with radius 4 m and volume 500 m$^3$? Use 3.14 for $\pi$ and round to the nearest tenth of a meter.

9. Some students volunteered to clean up a highway near their school. The amount of time it will take varies directly with the length of the section of highway and inversely with the number of students who will help. If 25 students clean up 5 mi of highway, the project will take 2 hr. How long would it take 85 students to clean up 34 mi of highway?

Write the function that models each variation. Find $z$ when $x = 2$ and $y = 6$.

10. $z$ varies inversely with $x$ and directly with $y$. When $x = 5$ and $y = 10$, $z = 2$.

11. $z$ varies directly with the square of $x$ and inversely with $y$. When $x = 2$ and $y = 4$, $z = 3$.

Each ordered pair is from an inverse variation. Find the constant of variation.

12. (2, 2)  
13. (1, 8)  
14. (9, 4)

Each pair of values is from an inverse variation. Find the missing value.

15. (9, 5), (x, 3)  
16. (8, 7), (5, y)  
17. (2, 7), (x, 1)
8-1 Enrichment
Inverse Variation

Each situation below can be modeled by a direct variation, inverse variation, joint variation, or combined variation equation. Decide which model to use and explain why.

1. The circumference $C$ of a circle is about 3.14 times the diameter $d$.

2. The number of cavities that develop in a patient’s teeth depends on the total number of minutes spent brushing.

3. The time it takes to build a bridge depends on the number of workers.

4. The number of minutes it will take to solve a problem set depends on the number of problems and the number of people working on the problem set.

5. The current $I$ in an electrical circuit decreases as the resistance $R$ increases.

6. Charles’s Gas Law states the volume $V$ of an enclosed gas at a constant pressure will increase as the absolute temperature $T$ increases.

7. Boyle’s Law states that the volume $V$ of an enclosed gas at a constant temperature is related to the pressure. The pressure of 3.45 L of neon gas is 0.926 atmosphere (atm). At the same temperature, the pressure of 2.2 L of neon gas is 1.452 atm.
Additional Vocabulary Support

The Reciprocal Function Family

For Exercises 1–5, draw a line from each word in Column A to its definition in Column B.

<table>
<thead>
<tr>
<th>Column A</th>
<th>Column B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. reciprocal</td>
<td>A. function that models inverse variation</td>
</tr>
<tr>
<td>2. branch</td>
<td>B. stretches, compressions, reflections, and horizontal and vertical translations</td>
</tr>
<tr>
<td>3. reciprocal function</td>
<td>C. multiplicative inverse</td>
</tr>
<tr>
<td>4. reflection of the reciprocal function</td>
<td>D. the graph of $y = \frac{1}{x}$</td>
</tr>
<tr>
<td>5. transformations</td>
<td>E. each part of the graph of a reciprocal function</td>
</tr>
</tbody>
</table>

For Exercises 6–9, the graph of each function is a transformation of the parent graph of $f(x) = \frac{1}{x}$. Draw a line from each function to its transformation.

<table>
<thead>
<tr>
<th></th>
<th>A. a horizontal translation</th>
<th>B. a reflection over the x-axis</th>
<th>C. a vertical translation</th>
<th>D. a stretch</th>
</tr>
</thead>
<tbody>
<tr>
<td>6. $f(x) = \frac{2}{x}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7. $f(x) = -\frac{1}{x}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8. $f(x) = \frac{1}{x-2}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9. $f(x) = \frac{1}{x} + 4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### A Reciprocal Function in General Form
The general form is \( y = \frac{a}{x - h} + k \), where \( a \neq 0 \) and \( x \neq h \).

The graph of this equation has a horizontal asymptote at \( y = k \) and a vertical asymptote at \( x = h \).

### Two Members of the Reciprocal Function Family
When \( a \neq 1 \), \( h = 0 \), and \( k = 0 \), you get the inverse variation function, \( y = \frac{a}{x} \).

When \( a = 1 \), \( h = 0 \), and \( k = 0 \), you get the parent reciprocal function, \( y = \frac{1}{x} \).

---

### Problem
What is the graph of the inverse variation function \( y = \frac{-5}{x} \)?

**Step 1** Rewrite in general form and identify \( a \), \( h \), and \( k \)

\[
y = \frac{-5}{x - 0} + 0 \quad a = 5, \; h = 0, \; k = 0
\]

**Step 2** Identify and graph the horizontal and vertical asymptotes.

- **Horizontal asymptote:** \( y = k \)
  \( y = 0 \)

- **Vertical asymptote:** \( x = h \)
  \( x = 0 \)

**Step 3** Make a table of values for \( y = \frac{-5}{x} \). Plot the points and then connect the points in each quadrant to make a curve.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-5</th>
<th>-2.5</th>
<th>-1</th>
<th>1</th>
<th>2.5</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>5</td>
<td>-2</td>
<td>-1</td>
</tr>
</tbody>
</table>

---

### Exercises
Graph each function. Include the asymptotes.

1. \( y = \frac{9}{x} \)
2. \( y = \frac{-4}{x} \)
3. \( xy = 2 \)
A reciprocal function in the form \( y = \frac{a}{x - h} + k \) is a translation of the inverse variation function \( y = \frac{a}{x} \). The translation is \( h \) units horizontally and \( k \) units vertically. The translated graph has asymptotes at \( x = h \) and \( y = k \).

**Problem**

What is the graph of the reciprocal function \( y = -\frac{6}{x + 3} + 2 \)?

**Step 1** Rewrite in general form and identify \( a, h, \) and \( k \).

\[
y = \frac{-6}{x - (-3)} + 2 \quad a = 6, \quad h = 3, \quad k = 2
\]

**Step 2** Identify and graph the horizontal and vertical asymptotes.

- Horizontal asymptote: \( y = k \)
- \( y = 2 \)
- Vertical asymptote: \( x = h \)
- \( x = 3 \)

**Step 3** Make a table of values for \( y = -\frac{6}{x} \), then translate each \((x, y)\) pair to \((x + h, y + k)\). Plot the translated points and connect the points in each quadrant to make a curve.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-6</th>
<th>-3</th>
<th>-2</th>
<th>2</th>
<th>3</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>-3</td>
<td>-2</td>
<td>-1</td>
</tr>
<tr>
<td>( x + (-3) )</td>
<td>-9</td>
<td>-6</td>
<td>-5</td>
<td>-1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>( y + 2 )</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

**Exercises**

Graph each function. Include the asymptotes.

4. \( y = \frac{3}{x - 2} - 4 \)
5. \( y = -\frac{4}{x - 8} \)
6. \( y = \frac{2}{3x} + \frac{3}{2} \)
Think About a Plan

The Reciprocal Function Family

a. **Gasoline Mileage** Suppose you drive an average of 10,000 miles each year. Your gasoline mileage (mi/gal) varies inversely with the number of gallons of gasoline you use each year. Write and graph a model for your average mileage \( m \) in terms of the gallons \( g \) of gasoline used.

b. After you begin driving on the highway more often, you use 50 gal less per year. Write and graph a new model to include this information.

c. Calculate your old and new mileage assuming that you originally used 400 gal of gasoline per year.

1. Write a formula for gasoline mileage in words.

2. Write and graph an equation to model your average mileage \( m \) in terms of the gallons \( g \) of gasoline used.

3. Write and graph an equation to model your average mileage \( m \) in terms of the gallons \( g \) of gasoline used if you use 50 gal less per year.

4. How can you find your old and your new mileage from your equations?

5. What is your old mileage?

6. What is your new mileage?
Complete this activity on your own.

**A Function Fable**

Given: \( g(x) = \frac{1}{x} \), \( s(x) = \frac{1}{x+2} \), \( d(x) = \frac{1}{x-3} \), \( m(x) = \frac{1}{x-3} + 6 \), \( p(x) = \frac{1}{x+2} + 3 \),

and \( f(x) = \frac{-1}{x+2} - 3 \)

Grandma function \( g(x) \) had two children. Her son Steve \( s(x) \) was left-handed and her daughter Diana \( d(x) \) was right-handed. Diana had one very tall child Michel \( m(x) \), who towered above her. Steve had two children as well. Pat \( p(x) \) and Jo \( f(x) \) were twins, but opposites of one another.

Graph the functions \( g(x) \), \( s(x) \), and \( p(x) \) on the grids below.

![Graphs](image-url)

**Activity**

Make a reciprocal function family with at least 3 "generations" and 6 individual functions. Explain the transformations that yield each member. Have at least one member in the third generation be a driving, graduate-athlete given:

- A horizontal translation corresponds to being a driver.
- A vertical translation corresponds to being an athlete.
- A reflection corresponds to being a high-school graduate.

*Note:* In the fable above, Jo was the only driving, graduate-athlete.

Then sketch a graph of one function from each generation (including the driving, graduate-athlete), showing all asymptotes.
Graph each function. Identify the x- and y-intercepts and the asymptotes of the graph. Also, state the domain and the range of the function.

1. \( y = \frac{12}{x} \)  
2. \( y = \frac{5}{x} \)  
3. \( y = -\frac{4}{x} \)

Use a graphing calculator to graph the equations \( y = \frac{1}{x} \) and \( y = \frac{a}{x} \) using the given value of \( a \). Then identify the effect of \( a \) on the graph.

4. \( a = 3 \)  
5. \( a = -5 \)  
6. \( a = 0.4 \)

Sketch the asymptotes and the graph of each function. Identify the domain and range.

7. \( y = \frac{1}{x} + 3 \)  
8. \( y = \frac{3}{4x} + \frac{1}{2} \)  
9. \( y = \frac{3}{x - 1} + 2 \)

Write an equation for the translation of \( y = -\frac{3}{x} \) that has the given asymptotes.

10. \( x = -1; y = 3 \)  
11. \( x = 4; y = -2 \)  
12. \( x = 0; y = 6 \)
13. The length of a pipe in a panpipe \( \ell \) (in feet) is inversely proportional to its pitch \( p \) (in hertz). The inverse variation is modeled by the equation \( p = \frac{497}{\ell} \). Find the length required to produce a pitch of 220 Hz.

Write each equation in the form \( y = \frac{k}{x} \).

14. \( y = \frac{4}{5x} \)

15. \( y = -\frac{7}{2x} \)

16. \( xy = -0.03 \)

Sketch the graph of each function.

17. \( xy = 6 \)

18. \( xy + 10 = 0 \)

19. \( 4xy = -1 \)

20. The junior class is buying keepsakes for Class Night. The price of each keepsake \( p \) is inversely proportional to the number of keepsakes \( s \) bought. The keepsake company also offers 10 free keepsakes in addition to the class's order. The equation \( p = \frac{1800}{s + 10} \) models this inverse variation.

a. If the class buys 240 keepsakes, what is the price for each one?

b. If the class pays $5.55 for each keepsake, how many can they get, including the free keepsakes?

c. If the class buys 400 keepsakes, what is the price for each one?

d. If the class buys 50 keepsakes, what is the price for each one?

Graph each pair of functions. Find the approximate point(s) of intersection.

21. \( y = \frac{3}{x - 4}; y = 2 \)

22. \( y = \frac{2}{x + 5}; y = -1.5 \)
Graph each function. Identify the $x$- and $y$-intercepts and asymptotes of the graph. Also, state the domain and range of the function.

1. $y = -\frac{2}{x}$
2. $y = \frac{4}{x}$
3. $y = -\frac{5}{x}$

Graph the equations $y = \frac{1}{x}$ and $y = \frac{a}{x}$ using the given value of $a$. Then identify the effect of $a$ on the graph.

4. $a = -3$
5. $a = 4$
6. $a = -0.25$

Sketch the asymptotes and the graph of each function. Identify the domain and range.

7. $y = \frac{1}{x} + 2$
8. $y = \frac{1}{x-2} + 3$
9. $y = \frac{-10}{x+1} - 8$
Practice (continued)

The Reciprocal Function Family

Write an equation for the translation of \( y = \frac{3}{x} \) that has the given asymptotes.

10. \( x = 0 \) and \( y = 2 \) 
11. \( x = -2 \) and \( y = 4 \) 
12. \( x = 5 \) and \( y = -3 \)

Sketch the graph of each function.

13. \( 3xy = 1 \) 
14. \( xy - 8 = 0 \) 
15. \( 2xy = -6 \)

16. **Writing** Explain the difference between what happens to the graph of the parent function of \( y = \frac{a}{x} \) when \( |a| > 1 \) and what happens when \( 0 < |a| < 1 \).

17. Suppose your class wants to get your teacher an end-of-year gift of a weekend package at her favorite spa. The package costs $250. Let \( c \) equal the cost each student needs to pay and \( s \) equal the number of students.
   a. If there are 22 students, how much will each student need to pay?
   b. Using the information, how many total students (including those from other classes) need to contribute to the teacher's gift, if no student wants to pay more than $7?
   c. **Reasoning** Did you need to round your answers up or down? Explain.
Understanding Horizontal Asymptotes

The line \( y = \frac{3}{4} \) is a horizontal asymptote for the graph of the function \( y = \frac{3x + 5}{4x - 8} \). By using long division, you can rewrite this function in the form quotient + remainder divided by the divisor: \( y = \frac{3}{4} + \frac{11}{4x - 8} \).

Examine what happens to the remainder divided by the divisor and the value of \( y \) as the value of \( x \) gets larger. Fill in the following table to four decimal places.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \frac{11}{4x - 8} )</th>
<th>( y = \frac{3}{4} + \frac{11}{4x - 8} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. 3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. 10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. 100</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note that as \( x \) gets larger, both the remainder and the value of \( y \) get smaller.

Although the value of \( y \) is always greater than \( \frac{3}{4} \), it gets closer to \( \frac{3}{4} \) as \( x \) gets larger. As \( x \) gets infinitely large, \( y \) approaches \( \frac{3}{4} \) from above. Write this as: As \( x \rightarrow +\infty \), \( y \rightarrow \frac{3}{4} \) from above.

Examine what happens as \( x \) gets smaller. Fill in the following table to four decimal places.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \frac{11}{4x - 8} )</th>
<th>( y = \frac{3}{4} + \frac{11}{4x - 8} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4. -3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. -10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. -100</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Here the value of \( y \) is always less than \( \frac{3}{4} \), but it gets closer to \( \frac{3}{4} \) as \( x \) gets smaller (more negative). Write this as: As \( x \rightarrow -\infty \), \( y \rightarrow \frac{3}{4} \) from below.

In both cases, \( y \) approaches \( \frac{3}{4} \), so the horizontal asymptote is \( y = \frac{3}{4} \).
Rational Functions and Their Graphs

A rational function may have one or more types of discontinuities: holes (removable points of discontinuity), vertical asymptotes (non-removable points of discontinuity), or a horizontal asymptote.

<table>
<thead>
<tr>
<th>If</th>
<th>Then</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$ is a zero with multiplicity $m$ in the numerator and multiplicity $n$ in the denominator, and $m \geq n$</td>
<td>hole at $x = a$</td>
<td>$f(x) = \frac{(x - 5)(x + 6)}{(x - 5)}$ hole at $x = 5$</td>
</tr>
<tr>
<td>$a$ is a zero of the denominator only, or $a$ is a zero with multiplicity $m$ in the numerator and multiplicity $n$ in the denominator, and $m &lt; n$</td>
<td>vertical asymptote at $x = a$</td>
<td>$f(x) = \frac{x^2}{x - 3}$ vertical asymptote at $x = 3$</td>
</tr>
</tbody>
</table>

Let $p = \text{degree of numerator}$.
Let $q = \text{degree of denominator}$.

- $m < n$  
  horizontal asymptote at $y = 0$  
  $f(x) = \frac{4x^2}{7x^2 + 2}$ horizontal asymptote at $y = \frac{4}{7}$
- $m > n$  
  no horizontal asymptote exists
- $m = n$  
  horizontal asymptote at $y = \frac{a}{b}$, where $a$ and $b$ are coefficients of highest degree terms in numerator and denominator

**Problem**

What are the points of discontinuity of $y = \frac{x^2 + x - 6}{3x^2 - 12}$, if any?

**Step 1** Factor the numerator and denominator completely. $y = \frac{(x - 2)(x + 3)}{3(x - 2)(x + 2)}$

**Step 2** Look for values that are zeros of both the numerator and the denominator. The function has a hole at $x = 2$.

**Step 3** Look for values that are zeros of the denominator only. The function has a vertical asymptote at $x = 2$.

**Step 4** Compare the degrees of the numerator and denominator. They have the same degree. The function has a horizontal asymptote at $y = \frac{1}{3}$

**Exercises**
Rational Functions and Their Graphs

Before you try to sketch the graph of a rational function, get an idea of its general shape by identifying the graph’s holes, asymptotes, and intercepts.

**Problem**

What is the graph of the rational function \( y = \frac{x + 3}{x + 1} \)?

**Step 1** Identify any holes or asymptotes.

- No holes; vertical asymptote at \( x = 1 \); horizontal asymptote at \( y = \frac{1}{1} = 1 \)

**Step 2** Identify any \( x \)- and \( y \)-intercepts.

- \( x \)-intercepts occur when \( y = 0 \). \( y \)-intercepts occur when \( x = 0 \).

\[
\begin{align*}
\frac{x + 3}{x + 1} &= 0 \\
x + 3 &= 0 \\
x &= -3 \\
\text{x-intercept at } -3
\end{align*}
\]

\[
\begin{align*}
y &= \frac{0 + 3}{0 + 1} \\
y &= 3 \\
\text{y-intercept at } 3
\end{align*}
\]

**Step 3** Sketch the asymptotes and intercepts.

**Step 4** Make a table of values, plot the points, and sketch the graph.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-1</td>
</tr>
<tr>
<td>-1.5</td>
<td>-3</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1.5</td>
</tr>
</tbody>
</table>

**Exercises**

Graph each function. Include the asymptotes.

4. \( y = \frac{4}{x^2 - 9} \)

5. \( y = \frac{x^2 + 2x - 2}{x - 1} \)
Think About a Plan
Rational Functions and Their Graphs

Grades A student earns an 82% on her first test. How many consecutive 100% test scores does she need to bring her average up to 95%? Assume that each test has equal impact on the average grade.

Understanding the Problem

1. One test score is __________.

2. The average of all the test scores is __________.

3. What is the problem asking you to determine?

Planning the Solution

4. Let \( x \) be the number of 100% test scores. Write an expression for the total number of test scores.

5. Write an expression for the sum of the test scores.

6. How can you model the student’s average as a rational function?

Getting an Answer

7. How can a graph help you answer this question?

8. What does a fractional answer tell you? Explain.

9. How many consecutive 100% test scores does the student need to bring her average up to 95%?
Game: Graphionary
Rational Functions and Their Graphs

This game is for teams of six to eight students. Each team will need graph paper.

Game Play

- Begin by cutting out the playing cards below. Shuffle them, then place them face down in a pile. Determine which team goes first.
- One player draws a card when it is his or her team’s turn. The player must sketch a detailed graph of the rational function found on the card. Team members must take turns being the sketcher.
- The rest of the team must try to come up with the equation of the rational function from the graph. Your teacher will determine how much time is allowed once the completed graph is shown to the team.
- Once the official guess is presented, the illustrator compares it to the function he or she drew. If correct, the team gets the points shown on the card. If incorrect, the opposing team can earn the points by giving the correct answer.
- The card goes back in the pile if no team gets the correct function.
- Play alternates until all cards have been used. The team with the most points wins.
8-3  Practice

Rational Functions and Their Graphs

Find the domain, points of discontinuity, and x- and y-intercepts of each rational function. Determine whether the discontinuities are removable or nonremovable.

1. \( y = \frac{(x - 4)(x + 3)}{x + 3} \)  
2. \( y = \frac{(x - 3)(x + 1)}{x - 2} \)

3. \( y = \frac{2}{x + 1} \)  
4. \( y = \frac{4x}{x^4 + 16} \)

Find the vertical asymptotes and holes for the graph of each rational function.

5. \( y = \frac{5 - x}{x^2 - 1} \)  
6. \( y = \frac{x^2 - 2}{x + 2} \)

7. \( y = \frac{x}{x(x - 1)} \)  
8. \( y = \frac{x + 3}{x^2 - 9} \)

9. \( y = \frac{x - 2}{(x + 2)(x - 2)} \)  
10. \( y = \frac{x^2 - 4}{x^2 + 4} \)

11. \( y = \frac{x^2 - 25}{x - 4} \)  
12. \( y = \frac{(x - 2)(2x + 3)}{(5x + 4)(x - 3)} \)

Find the horizontal asymptote of the graph of each rational function.

13. \( y = \frac{2}{x - 6} \)  
14. \( y = \frac{x + 2}{x - 4} \)  
15. \( y = \frac{2x^2 + 3}{x^3 - 6} \)  
16. \( y = \frac{3x - 12}{x^2 - 2} \)

Sketch the graph of each rational function.

17. \( y = \frac{3}{x - 2} \)  
18. \( y = \frac{3}{(x - 2)(x + 2)} \)  
19. \( y = \frac{x}{x^2 + 4} \)  
20. \( y = \frac{x + 2}{x - 1} \)
21. How many milliliters of 0.75% sugar solution must be added to 100 mL of 1.5% sugar solution to form a 1.25% sugar solution?

22. A soccer player has made 3 of his last 24 shots on goal, or 12.5%. How many more consecutive goals does he need to raise his shots-on-goal average to at least 20%?

23. Error Analysis A student listed the asymptotes of the function \( y = \frac{x^2 + 5x + 6}{x(x^2 + 4x + 4)} \) as shown at the right.

   Explain the student’s error(s). What are the correct asymptotes?

Sketch the graph of each rational function.

24. \( y = \frac{x}{x(x - 6)} \)  \hspace{1cm} 25. \( y = \frac{2x}{x - 6} \)  \hspace{1cm} 26. \( y = \frac{x^2 - 1}{x^2 - 4} \)  \hspace{1cm} 27. \( y = \frac{2x^2 + 10x + 12}{x^2 - 9} \)

28. You start a business word-processing papers for other students. You spend $3500 on a computer system and office furniture. You figure additional costs at $0.02 per page.

   a. Write a rational function modeling the total average cost per page.

      Graph the function.

   b. What is the total average cost per page if you type 1000 pages? If you type 2000?

   c. How many pages must you type to bring your total average cost to less than $1.50 per page?

   d. What are the vertical and horizontal asymptotes of the graph of the function?
8-3 Practice
Rational Functions and Their Graphs

Find the domain, points of discontinuity, and x- and y-intercepts of each rational function. Determine whether the discontinuities are removable or non-removable. To start, factor the numerator and denominator, if possible.

1. \( y = \frac{x+5}{x-2} \)
2. \( y = \frac{1}{x^2 + 2x + 1} \)
3. \( y = \frac{x+4}{x^2 + 2x - 8} \)

Find the vertical asymptotes and holes for the graph of each rational function.

4. \( y = \frac{x+6}{x+4} \)
5. \( y = \frac{(x - 2)(x - 1)}{x - 2} \)
6. \( y = \frac{x+1}{(3x - 2)(x - 3)} \)

Find the horizontal asymptote of the graph of each rational function. To start, identify the degree of the numerator and denominator.

7. \( y = \frac{x+1}{x+5} \)  \( \frac{x+1}{x+5} \rightarrow \text{degree 1} \)

8. \( y = \frac{x+2}{2x^2 - 4} \)  \( \frac{x+2}{2x^2 - 4} \rightarrow \text{degree 1} \)

9. \( y = \frac{3x^3 - 4}{4x + 1} \)

Sketch the graph of each rational function.

10. \( y = \frac{x+2}{(x+3)(x-4)} \)
11. \( y = \frac{x+3}{(x-1)(x-5)} \)
12. \( y = \frac{2x}{3x - 1} \)
8-3 Practice (continued)

Rational Functions and Their Graphs

13. The CD-ROMs for a computer game can be manufactured for $0.25 each. The development cost is $124,000. The first 100 discs are samples and will not be sold.
   a. Write a function for the average cost of a disc that is not a sample.
   b. What is the average cost if 2000 discs are produced? If 12,800 discs are produced?
   c. Reasoning How could you find the number of discs that must be produced to bring the average cost under $8?
   d. How many discs must be produced to bring the average cost under $8?

14. Error Analysis For the rational function \( y = \frac{x^2 - 2x - 8}{x^2 - 9} \), your friend said that the vertical asymptote is \( x = 1 \) and the horizontal asymptotes are \( y = 3 \) and \( y = -3 \). Without doing any calculations, you know this is incorrect. Explain how you know.

Sketch the graph of each rational function.

15. \( y = \frac{4x^2 - 100}{2x^2 + x - 15} \)
16. \( y = \frac{2x^2}{5x + 1} \)
17. \( y = \frac{2}{x^2 - 4} \)

18. Multiple Choice What are the points of discontinuity for the graph of

\[ y = \frac{(2x + 3)(x - 5)}{(x + 5)(2x - 1)} \]

A. -5, 1   B. -3/2, 5   C. -5, 1/2   D. 5, -1/2
8-3  Enrichment
Rational Functions and Their Graphs

Other Asymptotes
Recall that a rational function does not have a horizontal asymptote if the degree of the numerator is greater than the degree of the denominator. If, however, the degree of the numerator is exactly one more than the degree of the denominator, then the graph of the function has a slant asymptote.

You can use long division to find the equation of a slant asymptote. The equation of the slant asymptote is given by the quotient, disregarding the remainder.

1. Use long division to find the slant asymptote of \( f(x) = \frac{x^2 - x}{x + 1} \).

   The slant asymptote of the function is \( y = \underline{\phantom{000}} \).

2. Graph the slant asymptote on a coordinate grid.

3. Graph the vertical asymptote for the equation in Exercise 1 on the grid.

4. Copy and complete the table, and plot the points accordingly.

\[
\begin{array}{|c|c|c|c|c|}
\hline
x & -3 & -2 & 0 & 2 & 3 \\
\hline
f(x) & & & & & \\
\hline
\end{array}
\]

Connect with a smooth curve, being sure to draw near to all asymptotes.

5. Use long division to find the slant asymptote of \( f(x) = \frac{x^3}{x^2 + 1} \).

   The slant asymptote of the function is \( y = \underline{\phantom{000}} \).

6. Graph the slant asymptote on a new coordinate grid.

7. Find any vertical asymptotes for the equation in Exercise 5 and graph on the grid.

8. Copy and complete the table, and plot the points accordingly.

\[
\begin{array}{|c|c|c|c|c|}
\hline
x & -3 & -2 & 0 & 2 & 3 \\
\hline
f(x) & & & & & \\
\hline
\end{array}
\]

Connect with a smooth curve, being sure to draw near to all asymptotes.

9. The technique used to find slant linear asymptotes works for rational functions in which the degree of the numerator is one more than the degree of the denominator. Use long division to find the nonlinear asymptote of the rational function given by \( f(x) = \frac{x^3 + 1}{x} \).

   The asymptote of the function is \( y = \underline{\phantom{000}} \). Can you guess the shape of this asymptote?
There are two sets of cards that show how to simplify \( \frac{x^2 - 4}{x^2 - 2x + 1} \times \frac{x^2 + 2x - 3}{2x^2 - 3x - 2} \). The set on the left explains the thinking. The set on the right shows the steps. Write the steps in the correct order.

**Think Cards**
- Factor the numerators and denominators.
- Write the problem.
- Write the remaining factors.
- Divide out common factors.

**Write Cards**
- \( \frac{(x - 2)(x + 2)}{(x - 1)(x - 1)} \times \frac{(x + 3)(x - 1)}{(2x + 1)(x - 2)} \)
- \( \frac{(x + 2)(x + 3)}{(x - 1)(2x + 1)} \)
- \( \frac{x^2 - 4}{x^2 - 2x + 1} \times \frac{x^2 + 2x - 3}{2x^2 - 3x - 2} \)
- \( \frac{(x - 2)(x + 2)}{(x - 1)(x - 1)} \times \frac{(x + 3)(x - 1)}{(2x + 1)(x - 2)} \)

**Think**
- **First**, you should

**Write**
- **Step 1**
- **Step 2**
- **Step 3**
- **Step 4**
Rational Expressions

Simplest form of a rational expression means the numerator and the denominator have no factors in common. You may have to restrict certain values of the variable(s) when you write in simplest form, because division by zero is undefined.

Problem

What is the expression \( \frac{6x^3y^2 + 6x^2y^2 - 12xy^2}{3x^2y^3 - 12y^3} \) written in simplest form? State any restrictions on the variables.

\[
\frac{6x^2y^2(x^3 + x - 2)}{3y^3(x^2 - 4)}
\]

Factor \( 6x^2y^2 \) out of the numerator and \( 3y^3 \) out of the denominator.

\[
\frac{6x^2y^2(x^2 + x - 2)}{3y^3(x + 2)(x - 2)}
\]

Factor \( (x^2 + x - 2) \) and \( (x^2 - 4) \).

\[
\frac{(2 \cdot x \cdot y \cdot y)(x + 2)(x - 1)}{(x + 2)(x - 2)(x - 2)}
\]

Divide out the common factors.

\[
\frac{2x(x - 1)}{y(x - 2)}
\]

Write the remaining factors.

Look at the original expression.

\[
\frac{6x^2y^2(x + 2)(x - 1)}{3y^3(x + 2)(x - 2)}
\] is undefined if

\[
3y^3 = 0, x + 2 = 0, \text{ or } x - 2 = 0.
\]

So, \( y \neq 0, x \neq 2, \text{ and } x \neq 2. \)

Look at the simplified expression.

\[
\frac{2x(x - 1)}{y(x - 2)}
\] is undefined if

\[
y = 0 \text{ or } x = 2.
\]

So, \( y \neq 0 \text{ and } x \neq 2. \)

In simplest form, the expression is \( \frac{2x(x - 1)}{y(x - 2)} \), where \( y \neq 0, x \neq -2, \text{ and } x \neq 2. \)

Exercises

Simplify each rational expression. State any restrictions on the variable.

1. \( \frac{x^2 + x}{x^2 + 2x} \)

2. \( \frac{x^2 - 5x}{x^2 - 25} \)

3. \( \frac{x^2 + 3x - 18}{x^2 - 36} \)

4. \( \frac{4x^2 - 36}{x^2 + 10x + 21} \)

5. \( \frac{3x^2 - 12}{x^2 - x - 6} \)

6. \( \frac{x^2 - 9}{2x + 6} \)
Think About a Plan
Rational Expressions

Manufacturing A toy company is considering a cube or sphere-shaped container for packaging a new product. The height of the cube would equal the diameter of the sphere. Compare the ratios of the volumes to the surface areas of the containers. Which packaging will be more efficient? For a sphere, $SA = 4\pi r^2$.

Understanding the Problem

1. Let $x$ be the height of the cube. What are expressions for the cube’s volume and surface area?
   Volume: _____________ Surface area: _____________

2. Let $x$ be the diameter of the sphere. What are expressions for the sphere’s volume and surface area?
   Volume: _____________ Surface area: _____________

3. What is the problem asking you to do?

Planning the Solution

4. Write an expression for the ratio of the cube’s volume to its surface area. Simplify your expression.

5. Write an expression for the ratio of the sphere’s volume to its surface area. Simplify your expression.

Getting an Answer

6. Compare the ratios of the volumes to the surface areas of the containers. Which packaging will be more efficient?
Puzzle: Multiply and Conquer
Rational Expressions

Two rational expressions are shown below. Below each expression is a table with 27 rational expressions and polynomials.

1. Determine which of the 27 expressions you multiply by the original rational expression to get 1. Circle these expressions. One expression is circled below.

2. Then enter the bold numbers from the circled expressions into their corresponding locations on the board at the bottom of the page. The number 4 (shown in bold) is entered into the board in the block of cells corresponding to Expression A.

3. As an additional challenge, after completing Step 2, do the Sudoku puzzle. (Note: The objective is to fill the 9-by-9 grid so each column, row, and each of the nine 3-by-3 boxes contain the numbers 1 through 9 only one time each!)

Expression A: \[
\frac{(x^2 - 5x - 24)(x^2 - 5x - 36)}{(x^2 - 8x + 16)(x^2 + 11x + 30)(x^2 - 4)}
\]

<table>
<thead>
<tr>
<th>x+3</th>
<th>x-5</th>
<th>\frac{1}{x+2}</th>
<th>x-1</th>
<th>x-6</th>
<th>\frac{1}{x+4}</th>
<th>x-1</th>
<th>\frac{1}{x+6}</th>
<th>x-6</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-4</td>
<td>x+5</td>
<td>x-9</td>
<td>\frac{1}{x-7}</td>
<td>x-5</td>
<td>x-8</td>
<td>\frac{1}{x+5}</td>
<td>x+3</td>
<td>x+2</td>
</tr>
<tr>
<td>x+6</td>
<td>\frac{1}{x-8}</td>
<td>\frac{1}{x+3}</td>
<td>x-2</td>
<td>\frac{1}{x-3}</td>
<td>x+4</td>
<td>x-4</td>
<td>\frac{1}{x-9}</td>
<td>x-7</td>
</tr>
</tbody>
</table>

Expression B: \[
\frac{(3x^2 + 17x^2 + 20x)(2x^2 + 3x - 54)}{(3x^2 - 22x + 35)(2x^2 + 13x - 24)}
\]

<table>
<thead>
<tr>
<th>2x-7</th>
<th>3x-7</th>
<th>\frac{1}{x}</th>
<th>x-6</th>
<th>x-4</th>
<th>\frac{1}{2x-9}</th>
<th>\frac{1}{x+6}</th>
<th>\frac{1}{3x+5}</th>
<th>x+8</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-5</td>
<td>x+1</td>
<td>x-3</td>
<td>7x-1</td>
<td>\frac{1}{x-5}</td>
<td>\frac{1}{2x+3}</td>
<td>x+7</td>
<td>2x-3</td>
<td>\frac{1}{x+4}</td>
</tr>
<tr>
<td>\frac{1}{x+8}</td>
<td>\frac{1}{x+5}</td>
<td>3x-5</td>
<td>2x+9</td>
<td>2x-7</td>
<td>3x+4</td>
<td>\frac{1}{3x-7}</td>
<td>4x+1</td>
<td>\frac{1}{3x+2}</td>
</tr>
</tbody>
</table>
Simplify each rational expression. State any restrictions on the variables.

1. \( \frac{4x + 6}{2x + 3} \)
2. \( \frac{2y}{y^2 + 6y} \)
3. \( \frac{20 + 40x}{20x} \)
4. \( \frac{7x - 28}{x^2 - 16} \)
5. \( \frac{3y^2 - 3}{y^2 - 1} \)
6. \( \frac{3x^2 - 12}{x^2 - x - 6} \)
7. \( \frac{x^2 + 3x - 18}{x^2 - 36} \)
8. \( \frac{x^2 + 13x + 40}{x^2 - 2x - 35} \)

Multiply. State any restrictions on the variables.

9. \( \frac{5a}{5a + 5} \cdot \frac{10a + 10}{a} \)
10. \( \frac{2x + 4}{10x} \cdot \frac{15x^2}{x + 2} \)
11. \( \frac{x^2 - 5x}{x^2 + 3x} \cdot \frac{x + 3}{x - 5} \)
12. \( \frac{x^2 - 6x}{x^2 + 36} \cdot \frac{x + 6}{x^2} \)
13. \( \frac{5y - 20}{3y + 15} \cdot \frac{7y + 35}{10y + 40} \)
14. \( \frac{x - 2}{(x + 2)^2} \cdot \frac{x + 2}{2x - 4} \)
15. \( \frac{3x^3}{x^2 - 25} \cdot \frac{x^2 + 6x + 5}{x^2} \)
16. \( \frac{y^2 - 2y}{y^2 + 7y - 18} \cdot \frac{y^2 - 81}{y^2 - 11y + 18} \)

Divide. State any restrictions on the variables.

17. \( \frac{7x^4}{24y^5} \div \frac{21x}{12y^4} \)
18. \( \frac{6x + 6}{7} \div \frac{4x + 4}{x - 2} \)
19. \( \frac{5y}{2x^2} \div \frac{5y^3}{8x^3} \)
20. \( \frac{3y + 3}{6y + 12} \div \frac{18}{5y + 5} \)
21. \( \frac{y^2 - 49}{(y - 7)^2} \div \frac{5y + 35}{y^2 - 7y} \)
22. \( \frac{x^2 + 10x + 16}{x^2 - 6x - 16} \div \frac{x + 8}{x^2 - 64} \)
23. \( \frac{y^2 - 5y + 4}{y^2 - 1} \div \frac{y^2 - 9}{y^2 + 5y + 4} \)
24. \( \frac{x^2 - 4}{x^2 + 6x + 9} \div \frac{x^2 + 4x + 4}{x^2 - 9} \)
25. A farmer must decide whether to build a cylindrical grain silo or a rectangular grain silo. The cylindrical silo has radius \( r \). The rectangular silo has width \( r \) and length \( 2r \). Both silos have the same height \( h \).
   a. Write and simplify an expression for the ratio of the volume of the cylindrical silo to its surface area, including the circular floor and ceiling.
   b. Write and simplify an expression for the ratio of the volume of the rectangular silo to its surface area, including the rectangular floor and ceiling.
   c. Compare the ratios of volume to surface area for the two silos.
   d. Compare the volumes of the two silos.
   e. Reasoning Assume the average cost of construction materials per square foot of surface area is the same for either silo. How can you measure the cost-effectiveness of each silo?

Simplify each rational expression. State any restrictions on the variables.

26. \( \frac{2x^2 + 11x + 5}{3x^3 + 17x + 10} \)
   27. \( \frac{6x^2 + 5xy - 6y^2}{3x^2 - 5xy + 2y^2} \)

Multiply or divide. State any restrictions on the variables.

28. \( \frac{x^2 + 2x + 1}{x^2 - 1} \times \frac{x^2 + 3x + 2}{x^2 + 4x + 4} \)
   29. \( \frac{x^2 - 3x - 10}{2x^2 - 11x + 5} \div \frac{x^2 - 5x + 6}{2x^2 - 7x + 3} \)

30. Reasoning A rectangle has area \( \frac{10b}{6b - 6} \) and length \( \frac{b + 2}{2b - 2} \). Write an expression for the width of the rectangle.

31. Open-Ended Write three rational expressions that simplify to \( \frac{x + 1}{x - 1} \).
Simplify each rational expression. State any restrictions on the variables.

1. \[ \frac{-27x^3y}{9x^4y} \]

2. \[ \frac{-6 + 3x}{x^3 - 6x + 8} \]

3. \[ \frac{2x^2 - 3x - 2}{x^2 - 5x + 6} \]

Multiply. State any restrictions on the variables.

To start, factor all polynomials.

4. \[ \frac{4x^2 - 1}{2x^2 - 5x - 3} \times \frac{x^2 - 6x + 9}{2x^2 + 5x - 3} \]

\[ \frac{(2x + 1)(2x - 1)}{(2x + 1)(x - 3)(x + 3)} \]

5. \[ \frac{2x^2 + 7x + 3}{x - 4} \times \frac{x^2 - 16}{x^2 + 8x + 15} \]

6. \[ \frac{4x^2}{5y} \times \frac{7y}{12x^4} \]

Divide. State any restrictions on the variables.

To start, rewrite the division as multiplication by the reciprocal.

7. \[ \frac{16x^5}{3y^3} \div \frac{8x^3}{9y^3} \]

8. \[ \frac{x^2 + 2x - 15}{x^2 - 16} \div \frac{x + 1}{3x - 12} \]

9. \[ \frac{3y - 12}{2y + 4} \div \frac{6y - 24}{4y + 8} \]
10. Your school wants to build a courtyard surrounded by a low brick wall. It wants the maximum area for a given amount of brick wall. The courtyard can be either a circle or an equilateral triangle. Which shape would have the greater area to perimeter ratio?

Simplify each rational expression. State any restrictions on the variables.

11. \( \frac{x^2 - 2x - 8}{3x^2 + 4x - 4} \)

12. \( \frac{6x + 15}{2x^2 + 3x - 5} \)

13. \( \frac{x^2 - y^2}{6x^2 + 6xy} \)

14. **Writing** How can you tell whether a rational expression is in simplest form? Include an example with your explanation.

15. The width of a rectangle is given by the expression \( \frac{x + 10}{3x + 24} \) and the area can be represented by \( \frac{2x + 20}{6x + 15} \). What is the length of the rectangle?

16. **Multiple Choice** Which expression can be simplified to \( \frac{x - 1}{x - 3} \)?

   A. \( \frac{x^2 - x - 6}{x^2 - x - 2} \)

   B. \( \frac{x^2 - 2x + 1}{x^2 + 2x - 3} \)

   C. \( \frac{x^2 - 3x - 4}{x^2 - 7x + 12} \)

   D. \( \frac{x^2 - 4x + 3}{x^2 - 6x + 9} \)
8-4 Enrichment
Rational Expressions

In previous lessons, you learned how to graph reciprocal functions of the form
\[ f(x) = \frac{a}{x - h} + k \]. You learned that graphs of reciprocal functions have a
horizontal asymptote at \( y = k \) and a vertical asymptote at \( x = h \). For other types of
rational functions, the asymptotes are not as easily determined.

1. Explain why the reciprocal function \( f(x) = \frac{1}{x} \) is not the parent graph of
   \[ f(x) = \frac{x^2 + 2x - 3}{x^2 - 5x - 6} \].

2. For rational functions, vertical asymptotes are lines located at the value(s) of \( x \) that
   make the denominator 0. Write the equations of the vertical asymptotes for the
   rational function \( f(x) = \frac{x^2 + 2x - 3}{x^2 - 5x - 6} \).

3. Use a graphing calculator to graph the rational function \( f(x) = \frac{x^2 + 2x - 3}{x^2 - 5x - 6} \). Use your
   graph to check if your vertical asymptotes are correct. How does your graph confirm that the
   reciprocal function \( f(x) = \frac{1}{x} \) is not the parent graph?

4. What are the vertical asymptotes for the rational function \( f(x) = \frac{x^2 + 3x + 2}{x^2 - 5x + 6} \)?
   Confirm that these are the vertical asymptotes by graphing the function. What do you notice?

5. From the graph of the function in Exercise 4, you saw that even though \( x = 2 \) made
   the denominator equal 0, it was not an asymptote. A value such as this creates a
   hole in the graph, and is called a removable discontinuity. The value is not part of
   the domain, yet it is not a vertical asymptote. Factor the rational expression
   \( \frac{x^2 + 3x + 2}{x^2 - 5x + 6} \) and use your results to explain why \( x = 2 \) is not an asymptote.

6. For the rational function \( f(x) = \frac{2x^2 + 7x - 4}{x^2 + x - 20} \), algebraically determine the location of the
   vertical asymptote and the value at which there is a removable discontinuity.
### Additional Vocabulary Support

**Adding and Subtracting Rational Expressions**

The column on the left shows the steps used to subtract two rational expressions. Use the column on the left to answer each question in the column on the right.

<table>
<thead>
<tr>
<th>Problem</th>
<th>1. Read the problem. What process are you going to use to solve the problem?</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>What is the difference of the two rational expressions in simplest form? State any restrictions on the variable.</strong></td>
<td></td>
</tr>
<tr>
<td>[ \frac{2}{x^2 - 36} - \frac{1}{x^2 + 6x} ]</td>
<td></td>
</tr>
<tr>
<td><strong>Factor the denominators.</strong></td>
<td></td>
</tr>
<tr>
<td>[ \frac{2}{(x - 6)(x + 6)} - \frac{1}{x(x + 6)} ]</td>
<td></td>
</tr>
<tr>
<td><strong>Rewrite each expression with the LCD.</strong></td>
<td></td>
</tr>
<tr>
<td>[ \frac{2}{(x - 6)(x + 6)} \cdot \frac{x}{x} - \frac{1}{x(x + 6)} \cdot \frac{(x - 6)}{(x - 6)} ]</td>
<td></td>
</tr>
<tr>
<td><strong>Add the numerators. Combine like terms.</strong></td>
<td></td>
</tr>
<tr>
<td>[ \frac{2x}{x(x - 6)(x + 6)} - \frac{x - 6}{x(x + 6)(x - 6)} ]</td>
<td></td>
</tr>
<tr>
<td><strong>Divide out the common factors.</strong></td>
<td></td>
</tr>
<tr>
<td>[ \frac{x - 6}{x(x - 6)(x + 6)} ]</td>
<td></td>
</tr>
<tr>
<td><strong>The difference of the expressions is for ( x \neq 0, \ x \neq 6, \ x \neq 6 )</strong></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>2. Why do you factor the denominators?</th>
</tr>
</thead>
<tbody>
<tr>
<td>3. What does LCD stand for?</td>
</tr>
<tr>
<td>4. Why is the numerator ( x + 6 )?</td>
</tr>
<tr>
<td>5. Why is the numerator equal to 1?</td>
</tr>
<tr>
<td>6. Why are 0, 6, and 6 restrictions on the variable?</td>
</tr>
</tbody>
</table>
**Reteaching**

Adding and Subtracting Rational Expressions

Adding and subtracting rational expressions is a lot like adding and subtracting fractions. Before you can add or subtract the expressions, they must have a common denominator. The easiest common denominator to work with is the least common denominator, or LCD.

**Problem**

What is the LCD of \( \frac{6x}{x^3+2x^2} \) and \( \frac{5}{x^3+x^2-2x} \)?

\[
x^3 + 2x^2 = x^2(x + 2)
\]
\[
x^3 + x^2 - 2x = x(x^2 + x - 2) = x(x + 2)(x - 1)
\]

Completely factor each denominator.

Make a list of all the factors.

Cross off any repeated factors.

When the only difference between factors is the exponent (like \( x^2 \) and \( x \)), cross off all but the factor with the greatest exponent.

Multiply the remaining factors on the list. The product is the LCD.

The LCD of \( \frac{6x}{x^3+2x^2} \) and \( \frac{5}{x^3+x^2-2x} \) is \( x^2(x + 2)(x - 1) \).

**Exercises**

Assume that the polynomials given are the denominators of rational expressions. Find the LCD of each set.

1. \( x + 3 \) and \( 2x + 6 \)
2. \( 2x - 1 \) and \( 3x + 4 \)
3. \( x^2 - 4 \) and \( x + 2 \)
4. \( x^2 + 7x + 12 \) and \( x + 4 \)
5. \( x^2 + 5 \) and \( x - 25 \)
6. \( x^3 \) and \( 6x^2 \)
7. \( x, 2x, \) and \( 4x^3 \)
8. \( x^2 + 8x + 16 \) and \( x + 4 \)
9. \( x^2 + 4x - 5 \) and \( x^2 - x^2 \)
10. \( x^2 - 9 \) and \( x^2 + 2x - 3 \)
8-5  **Reteaching**  (continued)

**Adding and Subtracting Rational Expressions**

To find the sum or difference of rational expressions with unlike denominators:
- completely factor each denominator
- identify the least common denominator, or LCD
- multiply each expression by the factors needed to produce the LCD
- add or subtract numerators, and put the result over the LCD

**Problem**

What is the difference of \( \frac{2x}{3x^2 + 5x} - \frac{14}{3x^2 + 26x + 35} \) in simplest form? State any restrictions on the variable.

\[
\begin{align*}
3x^2 + 5x &= x(3x + 5) \\
3x^2 + 26x + 35 &= (3x + 5)(x + 7) \\
\frac{2x}{x(3x + 5)} \cdot \frac{(x + 7)}{(x + 7)} - \frac{14}{(3x + 5)(x + 7)} \cdot \frac{1}{1} \\
&= \frac{2x(x + 7)}{x(3x + 5)(x + 7)} - \frac{14x}{x(3x + 5)(x + 7)} \\
&= \frac{2x^2 + 14x - 14x}{x(3x + 5)(x + 7)} \\
&= \frac{2x}{3x^2 + 26x + 35}
\end{align*}
\]

Multiply to produce the LCD.

Completely factor each denominator.

Identify the LCD.

Subtract the numerators.

Distribute.

Simplify.

Therefore, \( \frac{2x}{3x^2 + 5x} - \frac{14}{3x^2 + 26x + 35} = \frac{2x}{3x^2 + 26x + 35} \), where \( x \neq -7, -\frac{5}{3}, 0 \).

**Exercises**

Simplify each sum or difference. State any restrictions on the variable.

11. \( \frac{y}{y - 1} + \frac{2}{1 - y} \)

12. \( \frac{3}{x + 2} + \frac{2}{x^2 - 4} \)

13. \( \frac{x}{x^2 + 5x + 6} - \frac{2}{x^2 + 3x + 2} \)

14. \( \frac{4x + 1}{x^2 - 4} - \frac{3}{x - 2} \)
Think About a Plan
Adding and Subtracting Rational Expressions

Optics To read small font, you use the magnifying lens with the focal length 3 in. How far from the magnifying lens should you place the page if you want to hold the lens at 1 foot from your eyes? Use the thin-lens equation.

Know
1. The focal length of the magnifying lens is .

2. The distance from the lens to your eyes is .

3. The thin-lens equation is .

Need
4. To solve the problem I need to find:

Plan
5. What variables in the thin-lens equation have values that are known?

6. Solve the thin-lens equation for the variable whose value is unknown.

7. Substitute the known values into your equation and simplify.

8. How far from the page should you hold the magnifying lens?
Activity: Graphing Calculator Check

Adding and Subtracting Rational Expressions

This activity can be done in groups of two or three students. Discuss each group’s results once everyone is finished.

Example: Use your graphing calculator to add the following rational expressions.

\[ \frac{x}{x + 4} + \frac{3}{x - 3} \]

Step 1 Enter \( Y_1 = \frac{x}{x + 4} \) Turn on the \( Y_1 \) function only. (You switch a function on or off by moving the cursor over the equals sign and pressing

Step 2 Enter \( Y_2 = \frac{3}{x - 3} \) ENTER. A highlighted equals sign means that a function is

Step 3 Enter \( Y_3 = Y_1 + Y_2 \). Turned on.) Graph \( Y_3 \) (see Figures 1 and 2).

\[ \frac{x}{x + 4} + \frac{3}{x - 3} = \frac{x(x - 3)}{(x + 4)(x - 3)} + \frac{3(x + 4)}{(x - 3)(x + 4)} \]

Step 4 Add

\[ = \frac{(x^2 - 3x) + (3x + 12)}{(x + 4)(x - 3)} = \frac{x^2 + 12}{(x + 4)(x - 3)} \]

Step 5 On another calculator, have a classmate enter \( Y_4 = \frac{x^2 + 12}{(x + 4)(x - 3)} \) (see Figures 3 and 4).

Since these graphs coincide, you can conclude the addition was performed correctly.

Repeat this process for the following expressions.

1. \( \frac{2}{x + 3} + \frac{2}{x - 3} \)

2. \( \frac{2}{x - 2} - \frac{2}{x + 2} \)

3. \( \frac{1}{x^2 - 4x} + \frac{x}{x^2 - 16} \)

4. \( \frac{10}{x^2 - 3x - 10} - \frac{10}{x^2 + 3x - 10} \)
8-5 Practice

Adding and Subtracting Rational Expressions

Find the least common multiple of each pair of polynomials.

1. $3x(x + 2)$ and $6x(2x - 3)$
2. $2x^2 - 8x + 8$ and $3x^2 + 27x - 30$
3. $4x^3 + 12x + 9$ and $4x^2 - 9$
4. $2x^2 + 18$ and $5x^3 + 30x^2 + 45x$

Simplify each sum or difference. State any restrictions on the variables.

5. $\frac{x^2}{5} + \frac{x^2}{5}$
6. $\frac{6y - 4}{y^2 - 5} + \frac{3y + 1}{y^2 - 5}$
7. $\frac{2y + 1}{3y} + \frac{5y + 4}{3y}$
8. $\frac{12}{xy^3} - \frac{9}{xy^3}$
9. $\frac{2}{n + 4} - \frac{n^2}{n^2 - 16}$
10. $\frac{3}{8x^3y^3} - \frac{1}{4xy}$
11. $\frac{6}{5x^2y} + \frac{5}{10xy^2}$
12. $\frac{x + 2}{x^2 + 4x + 4} + \frac{2}{x + 2}$
13. $\frac{4}{x^3 - 25} + \frac{6}{x^3 + 6x + 5}$
14. $\frac{y}{4y + 8} - \frac{1}{y^2 + 2y}$

Simplify each complex fraction.

15. $\frac{2}{\frac{x}{3} - \frac{3}{y}}$
16. $\frac{1 + \frac{2}{x}}{4 - \frac{6}{x}}$
17. $\frac{1}{\frac{x - 2}{2} + \frac{1}{x}}$
18. $\frac{3}{\frac{x + 1}{5} - \frac{x - 1}{5}}$
19. $\frac{4}{\frac{x^2 - 1}{3} - \frac{x + 1}{x + 1}}$
20. $\frac{1 + \frac{2}{3}}{\frac{4}{9}}$
21. $\frac{\frac{2}{x} + 6}{\frac{1}{y}}$
22. $\frac{\frac{x - 3}{x^3 - 9}}{\frac{x^3 - 9}{3x - 9}}$
23. $\frac{\frac{5}{x + 3}}{2 + \frac{1}{x + 3}}$
24. The total resistance for a parallel circuit is given by: \( \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \).

a. If \( R = 1 \) ohm, \( R_2 = 6 \) ohms, and \( R_3 = 8 \) ohms, find \( R_1 \).

b. If \( R_1 = 3 \) ohms, \( R_2 = 4 \) ohms, and \( R_3 = 6 \) ohms, find \( R \).

Add or subtract. Simplify where possible. State any restrictions on the variables.

25. \( \frac{3}{7x^2y} + \frac{4}{21xy^2} \)

26. \( \frac{xy - y}{x - y} - \frac{y}{x + 2} \)

27. \( \frac{3}{x^2 - x - 6} + \frac{2}{x^2 + 6x + 5} \)

28. \( \frac{6}{y^2 + 5y} + \frac{3y}{4y + 20} - \frac{1}{4} \)

29. A teacher uses an overhead projector with a focal length of \( x \) cm. She sets a transparency \( x + 20 \) cm below the projector’s lens. Write an expression in simplest form to represent how far from the lens she should place the screen to place the image in focus. Use the thin-lens equation \( \frac{1}{f} = \frac{1}{d_i} + \frac{1}{d_o} \).

30. Open-Ended Write two complex fractions that simplify to \( \frac{x + 5}{x^2} \).

31. Writing Explain the differences in the process of adding two rational expressions using the lowest common denominator (LCD) and adding them using a common denominator that is not the LCD. Include an example in your explanation.
Find the least common multiple of each pair of polynomials.

To start, completely factor each expression.

1. \(4x^2 - 36\) and \(6x^3 + 36x + 54\)
   
   \((2)(2)(x - 3)(x + 3)\) and \((2)(3)(x + 3)(x + 3)\)

2. \((x - 2)(x + 3)\) and \(10(x + 3)^2\)

Simplify each sum or difference. State any restrictions on the variables.

To start, factor the denominators and identify the LCD.

3. \(\frac{6x - 1}{x^2y} + \frac{3y + 2}{2xy}\)
   
   \(\frac{6x - 1}{(x)(x)(y)} + \frac{3y + 2}{(2)(x)(y)}\)

4. \(\frac{1}{x^2 - 4x - 12} - \frac{3x}{4x + 8}\)

5. \(\frac{2x}{x^2 + 5x + 4} + \frac{2x}{3x + 3}\)

Add or subtract. Simplify where possible. State any restrictions on the variables.

6. \(\frac{x + 2}{x - 1} + \frac{x - 3}{2x + 1}\)

7. \(\frac{x}{x^2 - x} + \frac{1}{x}\)

8. \(4y - \frac{y + 2}{y^2 + 3y}\)

9. **Error Analysis** A classmate said that the sum of \(\frac{4}{x^2 - 9}\) and \(\frac{7}{x + 3}\) is \(\frac{7x + 25}{x^2 - 9}\)

   What mistake did your classmate make? What is the correct sum?
Practce (continued) 

Adding and Subtracting Rational Expressions

Simplify each complex fraction.

To start, multiply the numerator and the denominator by the LCD of all the rational expressions.

\[
\begin{align*}
10. \quad & \frac{\frac{1}{x} + 3}{\frac{5}{y} + 4} \\
11. \quad & \frac{-\frac{3}{5}}{\frac{1}{x} + y} \\
12. \quad & \frac{\frac{4}{x + 2}}{\frac{3}{x - 1}} \\
\end{align*}
\]

\[
\frac{(-3)xy}{(-4)xy}
\]

13. **Reasoning** What real numbers are not in the domain of the function \( f(x) = \frac{x + 1}{x + 3} \)?

Explain.

14. If you jog 12 mi at an average rate of 4 \( \text{mi/h} \) and walk the same route back at an average rate of 3 \( \text{mi/h} \), you have traveled 24 mi in 7 h and your overall rate is \( \frac{24}{7} \) mi/h. What is your overall average rate if you travel \( d \) mi at 3 \( \text{mi/h} \) and \( d \) mi at 4 \( \text{mi/h} \)?

15. **Multiple Choice** Simplify: \( \frac{\frac{2}{x} - \frac{5}{6}}{\frac{x}{6} - 3} \).

\[
\begin{align*}
A \quad & \frac{2 - 5x}{6 - 3x} \\
B \quad & \frac{2 + 5x}{6 - 3x} \\
C \quad & \frac{2 - 5x}{6x + 3} \\
D \quad & \frac{6 + 3x}{2 - 5x}
\end{align*}
\]
The Superposition Principle

The illumination received from a light source is given by the formula

\[ I = S \cdot D^{-2} \]  

where \( I \) is the illumination at a certain point, \( S \) is the strength of the light source, measured in watts or kilowatts, and \( D \) is the distance of the point from the light source. The superposition principle states that the total illumination received at a given point from two sources is equal to the sum of the illuminations from each of the sources.

Suppose a plant is positioned at point \( A \). Copy and complete the following to find the total illumination received by the plant when both lights are on.

\[
I_{\text{total}} = I_{L1} + I_{L2} \\
= \frac{100}{2^2} + \frac{200}{1^2} \\
= \text{_____} \\
= \text{_____} \quad \text{Round to the nearest tenth.}
\]

1. The amount of illumination received by the plant is \( \text{_____} \).

Lighthouse A, located on an ocean shore, uses a 10-kW light. Lighthouse B uses a 20-kW light and is located 8 km out to sea from a point 6 km down the beach from Lighthouse A.

2. A man is walking down the beach away from lighthouse A and toward point C. When he is \( x \) kilometers away from lighthouse A and has not yet reached point C, write the illumination he receives as a function of \( x \) in simplest form.

3. Now suppose that the man is \( x \) kilometers beyond point C as he walks down the beach. What illumination does he receive, written as a function of \( x \) in simplest form?
Additional Vocabulary Support

Problem

What are the solutions of the rational equation? Justify your steps.

\[
\frac{x}{x-2} + \frac{1}{x-4} = \frac{2}{x^2 - 6x + 8}
\]

Write original equation.

\[
\frac{x}{x-2} + \frac{1}{x-4} = \frac{2}{(x-2)(x-4)}
\]

Factor the denominators to find the LCD.

\[
(x-2)(x-4) \left[ \frac{x}{x-2} + \frac{1}{x-4} \right]
\]

Multiply each side by the LCD to clear the denominators.

\[
x(x-4) + 1(x-2) = 2
\]

Distribute and simplify.

\[
x^2 - 4x + x - 2 = 2
\]

Distribute.

\[
x^2 - 3x - 4 = 0
\]

Simplify.

\[
(x-4)(x+1) = 0
\]

Factor the quadratic.

\[
x = 4 \text{ or } x = -1
\]

Solve for x.

\[
x = 4 \text{ causes division by 0, so } x = 4 \text{ is an extraneous solution. Check for extraneous solutions.}
\]

\[
\frac{-1}{-1-2} + \frac{1}{-1-4} = \frac{2}{(-1)^2 - 6(-1) + 8'}
\]

Because \[
\text{the solution is } x = -1.
\]

Exercise

What are the solutions of the rational equation? Justify the steps.

\[
\frac{5}{x} + \frac{4}{x+3} = \frac{8}{x^2 + 3x}
\]

______________.

\[
\frac{5}{x} + \frac{4}{x+3} = \frac{8}{x(x+3)}
\]

______________.

\[
x(x+3) \left[ \frac{5}{x} + \frac{4}{x+3} \right] = x(x+3) \left[ \frac{8}{x(x+3)} \right]
\]

______________.

\[
9x + 15 = 8
\]

______________.

\[
x = \frac{-7}{9}
\]

______________.
8-6  Reteaching

When one or both sides of a rational equation has a sum or difference, multiply each side of the equation by the LCD to eliminate the fractions.

**Problem**

What is the solution of the rational equation \( \frac{6}{x} + \frac{x}{2} = 4 \)? Check the solutions.

\[
2x \left( \frac{6}{x} + \frac{x}{2} \right) = 2x(4) \quad \text{Multiply each term on both sides by the LCD, } 2x.
\]

\[
2x \left( \frac{6}{x} \right) + 2x \left( \frac{x}{2} \right) = 2x(4) \quad \text{Divide out the common factors.}
\]

\[
x^2 - 8x + 12 = 0 \quad \text{Simplify.}
\]

\[
(x - 2)(x - 6) = 0 \quad \text{Write the equation in standard form.}
\]

\[
x - 2 = 0 \text{ or } x - 6 = 0 \quad \text{Factor.}
\]

\[
x = 2 \text{ or } x = 6 \quad \text{Use the Zero-Product Property.}
\]

\[
x = 2 \text{ or } x = 6 \quad \text{Solve for } x.
\]

**Check**

\[
\frac{6}{x} + \frac{x}{2} = 4 \quad \frac{6}{x} + \frac{x}{2} \neq 4
\]

\[
\frac{6}{2} + \frac{2}{2} \neq 4 \quad \frac{6}{6} + \frac{6}{2} \neq 4
\]

\[
3 + 1 \neq 4 \quad 1 + 3 \neq 4
\]

\[
4 = 4 \checkmark \quad 4 = 4 \checkmark
\]

The solutions are \( x = 2 \) and \( x = 6 \).

**Exercises**

Solve each equation. Check the solutions.

1. \( \frac{10}{x+3} + \frac{10}{3} = 6 \)
2. \( \frac{1}{x-3} = \frac{x-4}{x^2-27} \)
3. \( \frac{6}{x-1} + \frac{2x}{x-2} = 2 \)
4. \( \frac{7}{3x-12} - \frac{1}{x-4} = \frac{2}{3} \)
5. \( \frac{2x}{5} = \frac{x^2-5x}{5x} \)
6. \( \frac{8(x-1)}{x^2-4} = \frac{4}{x-2} \)
7. \( \frac{x+4}{x} = \frac{25}{6} \)
8. \( \frac{2}{x} + \frac{6}{x-1} = \frac{6}{x^2-x} \)
9. \( \frac{2}{x} + \frac{1}{x} = 3 \)
10. \( \frac{4}{x-1} = \frac{5}{x-1} + 2 \)
11. \( \frac{1}{x} = \frac{5}{2x} + 3 \)
12. \( \frac{x+6}{5} = \frac{2x-4}{5} - 3 \)
You often can use rational equations to model and solve problems involving rates.

**Problem**

Quinn can refinish hardwood floors four times as fast as his apprentice, Jack. They are refinishing 100 ft² of flooring. Working together, Quinn and Jack can finish the job in 3 h. How long would it take each of them working alone to refinish the floor?

Let \( x \) be Jack’s work rate in ft²/h. Quinn’s work rate is four times faster, or 4\( x \).

\[
\text{square feet refinished per hour by } \quad \frac{\text{square feet of floor}}{\text{hours worked}} = \frac{\text{they refinish together}}{\text{together}}
\]

\[
x + 4x = \frac{100}{3}
\]

Their work rates sum to 100 ft² in 3 h.

\[
3(x) + 3(4x) = 3 \left( \frac{100}{3} \right)
\]

They work for 3 h. Refinished floor area = rate \( \times \) time.

\[
15x = 100
\]

Simplify.

\[
x \approx 6.67
\]

Divide each side by 15.

Jack works at the rate of 6.67 ft²/h. Quinn works at the rate of 26.67 ft²/h.

Let \( j \) be the number of hours Jack takes to refinish the floor alone, and let \( q \) be the number of hours Quinn takes to refinish the floor alone.

\[
6.67 = \frac{100}{j} \quad 26.67 = \frac{100}{q}
\]

\[
j(6.67) = j \left( \frac{100}{j} \right) \quad q(26.67) = q \left( \frac{100}{q} \right)
\]

\[
6.67j = 100 \quad 26.67q = 100
\]

\[
j = 15 \quad q \approx 3.75
\]

Jack would take 15 h and Quinn would take 3.75 h to refinish the floor alone.

**Exercises**

13. An airplane flies from its home airport to a city and back in 5 h flying time. The plane travels the 720 mi to the city at 295 mi/h with no wind. How strong is the wind on the return flight? Is the wind a headwind or a tailwind?

14. Miguel can complete the decorations for a school dance in 5 days working alone. Nasim can do it alone in 3 days, and Denise can do it alone in 4 days. How long would it take the three students working together to decorate?
Think About a Plan
Solving Rational Equations

**Storage** One pump can fill a tank with oil in 4 hours. A second pump can fill the same tank in 3 hours. If both pumps are used at the same time, how long will they take to fill the tank?

**Understanding the Problem**

1. How long does it take the first pump to fill the tank?

2. How long does it take the second pump to fill the tank?

3. What is the problem asking you to determine?

**Planning the Solution**

4. If \( V \) is the volume of the tank, what expressions represent the portion of the tank that each pump can fill in one hour?

   First pump: \( \frac{V}{4} \)  
   Second pump: \( \frac{V}{3} \)

5. What expression represents the part of the tank the two pumps can fill in one hour if they are used at the same time?

6. Let \( t \) be the number of hours. Write an equation to find the time it takes for the two pumps to fill one tank.

**Getting an Answer**

7. Solve your equation to find how long the pumps will take to fill the tank if both pumps are used at the same time.
Solve each equation. Check each solution.

1. \( \frac{x}{3} + \frac{x}{2} = 10 \)
2. \( \frac{1}{x} - \frac{x}{9} = 0 \)
3. \( \frac{-4}{x+1} = \frac{5}{3x+1} \)

4. \( \frac{4}{x} = \frac{x}{4} \)
5. \( \frac{3x}{4} = \frac{5x + 1}{3} \)
6. \( \frac{3}{2x - 3} = \frac{1}{5 - 2x} \)

7. \( \frac{x - 4}{3} = \frac{x - 2}{2} \)
8. \( \frac{2x - 1}{x + 3} = \frac{5}{3} \)
9. \( \frac{2y + 2}{5} = \frac{y - 1}{6} \)

10. \( \frac{1}{2x + 2} + \frac{5}{x^2 - 1} = \frac{1}{x - 1} \)
11. \( \frac{2}{x + 3} + \frac{5}{3 - x} = \frac{6}{x^2 - 9} \)

12. An airplane flies from its home airport to a city 510 mi away and back. The total flying time for the round-trip flight is 3.9 h. The plane travels the first half of the trip at 255 mi/h with no wind.
   a. How strong is the wind on the return flight? Round your answer to the nearest tenth.
   b. Is the wind on the return flight a headwind or a tailwind?

Use a graphing calculator to solve each equation. Check each solution.

13. \( \frac{x - 1}{6} = \frac{x}{4} \)
14. \( \frac{x - 2}{10} = \frac{x - 7}{5} \)
15. \( \frac{4}{x + 3} = \frac{10}{2x - 1} \)

16. \( \frac{3}{3 - x} = \frac{4}{2 - x} \)
17. \( \frac{3y}{5} + \frac{1}{2} = \frac{y}{10} \)
18. \( 5 - \frac{4}{x + 1} = 6 \)

19. \( \frac{2}{3} + \frac{3x - 1}{6} = \frac{5}{2} \)
20. \( \frac{4}{x - 1} = \frac{5}{x - 2} \)
21. \( \frac{1}{x} - \frac{2}{x + 3} = 0 \)

Solve each equation for the given variable.

22. \( h = \frac{24}{b} \); \( b \)
23. \( \frac{1}{f} = \frac{1}{d_i} = \frac{1}{d_o} \)

24. \( \frac{h}{t} + 16t = v_o \); \( h \)
25. \( m = \frac{y_2 - y_1}{x_2 - x_1} \); \( x_1 \)

26. \( \frac{xy}{z} + 2x = \frac{z}{y} \); \( x \)
27. \( \frac{S - 2wh}{2w + 2h} = l \); \( S \)
28. One delivery driver can complete a route in 6 h. Another driver can complete the same route in 5 h.
   a. Let \( N \) be the total number of deliveries on the route. Write expressions to represent the number of deliveries each driver can make in 1 hour.
   b. Write an expression to represent the number of hours needed to make \( N \) deliveries if the drivers work together.
   c. If the drivers work together, about how many hours will they take to complete the route? Round your answer to the nearest tenth.

29. A fountain has two drainage valves. With the first valve open, the fountain drains completely in 4 h. With only the second valve open, the fountain drains completely in 5.25 h. About how many hours will the fountain take to drain with both valves open? Round your answer to the nearest tenth.

30. A pen factory has two machines making pens. Together, the machines make 1500 pens during an 8-h shift. Machine A makes pens at 2.5 times the rate of Machine B. About how many hours would Machine A need to make 1500 pens by itself? Round your answer to the nearest tenth.

31. Error Analysis Describe and correct the error made in solving the equation.

32. The formula \( V = hH \left( \frac{b_1 + b_2}{6} \right) \) gives the volume of a pyramid with a trapezoidal base.
   a. Solve this equation for \( b_2 \).
   b. Find \( b_2 \) if \( b_1 = 5 \text{ cm, } h = 8 \text{ cm, } H = 9 \text{ cm, and } V = 216 \text{ cm}^3 \).
Solve each equation. Check each solution.

To start, multiply each side by the LCD.

1. \( \frac{x}{4} - \frac{3}{x} = \frac{1}{4} \)
2. \( x + \frac{6}{x} = -5 \)
3. \( \frac{5}{2x - 2} = \frac{15}{x^2 - 1} \)

4. The aerodynamic covering on a bicycle increases a cyclist’s average speed by 10 mi/h. The time for a 75-mi trip is reduced by 2 h.
   a. Using \( t \) for time, write a rational equation you can use to determine the average speed using the aerodynamic covering.
   b. What is the average speed for the trip using the aerodynamic covering?

Using a graphing calculator, solve each equation. Check each solution.

5. \( \frac{4}{2x - 3} = \frac{x}{5} \)
6. \( x + 5 = \frac{6}{x} \)
7. \( \frac{2}{x + 7} = \frac{x}{x^2 - 49} \)

Solve each equation for the given variable.

8. \( F = \frac{mv^2}{r} \) for \( v \)
9. \( \frac{c}{dt} = Qm \) for \( d \)
10. \( \frac{F}{Gm_1} = \frac{m_2}{r^2} \) for \( r \)
11. You can travel 40 mi on your motorbike in the same time it takes your friend to travel 15 mi on his bicycle. If your friend rides his bike 20 mi/h slower than you ride your motorbike, find the speed for each bike.

12. A passenger train travels 392 mi in the same time that it takes a freight train to travel 322 mi. If the passenger train travels 20 mi/h faster than the freight train, find the speed of each train.

13. You can paint a fence twice as fast as your sister can. Working together, the two of you can paint a fence in 6 h. How many hours would it take each of you working alone?

Solve each equation. Check each solution.

14. \[ \frac{2}{x - 3} - \frac{4}{x + 3} = \frac{8}{x^2 - 9} \]

15. \[ \frac{3}{x + 5} + \frac{2}{5 - x} = \frac{-4}{x^2 - 25} \]

16. \[ \frac{3}{x^2 - 1} + \frac{4x}{x + 1} = \frac{1.5}{x - 1} \]

17. You are planning a school field trip to a local theater. It costs $60 to rent the bus. Each theater ticket costs $5.50.
   a. Write a function \( c(x) \) to represent the cost per student if \( x \) students sign up for the trip.
   b. How many students must sign up if the cost is to be no more than $10 per student?
Gravitational Attraction

Many physical phenomena obey inverse-square laws. That is, the strength of the quantity is inversely proportional to the square of the distance from the source.

Isaac Newton was the first to discover that gravity obeys an inverse-square law. The gravitational force $F$ between objects of masses $M$ and $m$ separated by a distance $D$ is given by $F = \frac{GMm}{D^2}$, where $G$ is a constant.

Suppose that two stars, Alpha Major and Beta Minor, are separated by a distance of 6 light-years. Alpha Major has four times the mass of Beta Minor. Let $M$ represent the mass of Beta Minor. Suppose that an object, represented by point $P$, of mass $m$ is placed between the two stars at a distance of $D$ light-years from Beta Minor.

1. Write an expression for the gravitational force between this object and Beta Minor.

2. Write an expression for the gravitational force between this object and Alpha Major.

3. What is the distance of a neutral position of the object $P$ with mass $m$ from Beta Minor? At neutral position, both Beta Minor and Alpha Major exert equal force on point $P$.

A spaceship is stationary between a planet and its moon, experiencing an equal gravitational pull from each. When measurements are taken, it is determined that the craft is 300,000 km from the planet and 100,000 km from the moon.

4. What is the ratio of the mass of the planet to the mass of the moon?

5. What would be the ratio of their masses if the distance of the spaceship from the planet was $R$ times the distance of the spaceship to the moon?

Once every 277 yr, the two moons of the planet Omega Minus line up in a straight line with the planet. The moons are equal in mass, and the inner moon is equidistant from the outer moon and from the planet. Measurements show that an object two thirds of the distance from the planet to the inner moon, and in the same line as all three, experiences an equal gravitational pull in both directions.

6. What is the ratio of the mass of the planet to the mass of one of its moons?