THE SCHOOL DISTRICT OF PHILADELPHIA

Geometry

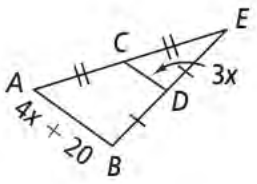
Answer Keys

THE SCHOOL DISTRICT OF PHILADELPHIA

5-1 Additional Vocabulary Support

Midsegments of Triangles

There are two sets of note cards below that show how to find *AB* and *CD* for the triangle at the right. The set on the left explains the thinking. The set on the right shows the steps. Write the thinking and the steps in the correct order.



Think Cards

Use x to find AB and CD.

Subtract 2x from each side.

Simplify the right side of the equation.

Substitute expressions for the length of each side.

Identify $\overline{CD}$ as the midsegment of $\triangle AEB$. Use the Triangle Midsegment Theorem.

Write Cards

$CD = \frac{1}{2}AB$

$3x - 2x = 2x + 10 - 2x$

$x = 10$

$3x = 2x + 10$

$AB = 4(10) + 20 = 60$

$CD = 3(10) = 30$

$3x = \frac{1}{2}(4x + 20)$

Think

First, you should **identify $\overline{CD}$ as the midsegment of $\triangle AEB$. Use the Triangle Midsegment Theorem.**

Second, you should **substitute expressions for the length of each side.**

Third, you should **simplify the right side of the equation.**

Next, you should **subtract 2x from each side.**

Finally, you should **use x to find AB and CD.**

Write

Step 1 $CD = \frac{1}{2}AB$

Step 2 $3x = \frac{1}{2}(4x + 20)$

Step 3 $3x = 2x + 10$

Step 4 $3x - 2x = 2x + 10 - 2x$
$x = 10$

Step 5 $AB = 4(10) + 20 = 60$
$CD = 3(10) = 30$

5-1 Reteaching

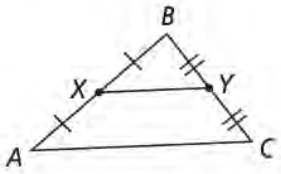
Midsegments of Triangles

Connecting the midpoints of two sides of a triangle creates a segment called a *midsegment* of the triangle.

Point X is the midpoint of $\overline{AB}$.

Point Y is the midpoint of $\overline{BC}$.

So, $\overline{XY}$ is a midsegment of $\triangle ABC$.

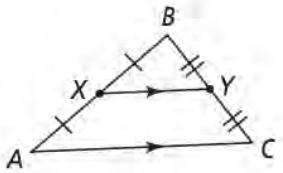


There is a special relationship between a midsegment and the side of the triangle that is not connected to the midsegment.

Triangle Midsegment Theorem

- The midsegment is parallel to the third side of the triangle.
- The length of the midsegment is half the length of the third side.

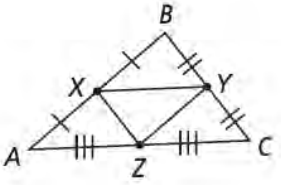
$\overline{XY} \parallel \overline{AC}$ and $XY = \frac{1}{2}AC$.



Connecting each pair of midpoints, you can see that a triangle has three midsegments.

$\overline{XY}$, $\overline{YZ}$, and $\overline{ZX}$ are all midsegments of $\triangle ABC$.

Because Z is the midpoint of $\overline{AC}$, $XY = AZ = ZC = \frac{1}{2}AC$.

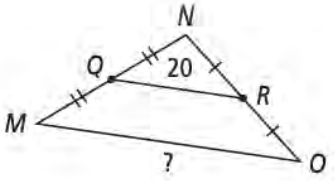


Problem

$\overline{QR}$ is a midsegment of $\triangle MNO$.

What is the length of $\overline{MO}$?

Start by writing an equation using the Triangle Midsegment Theorem.



$$\frac{1}{2}MO = QR$$

$$MO = 2QR$$

$$= 2(20)$$

$$= 40$$

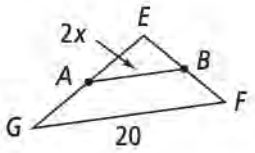
So, $MO = 40$.

5-1 Reteaching (continued)

Midsegments of Triangles

Problem

$\overline{AB}$ is a midsegment of $\triangle GEF$. What is the value of x?

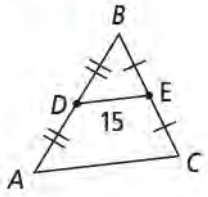


$$2AB = GF$$
$$2(2x) = 20$$
$$4x = 20$$
$$x = 5$$

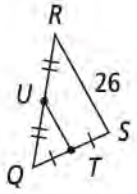
Exercises

Find the length of the indicated segment.

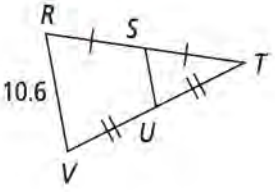
1. AC **30**



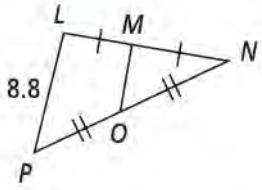
2. TU **13**



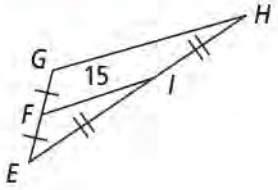
3. SU **5.3**



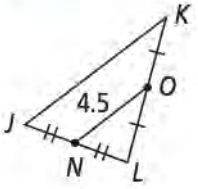
4. MO **4.4**



5. GH **30**

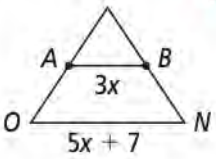


6. JK **9**

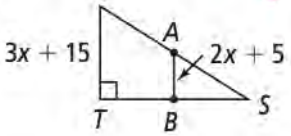


Algebra In each triangle, $\overline{AB}$ is a midsegment. Find the value of x.

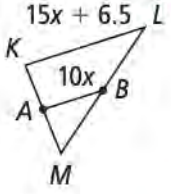
7. **7**



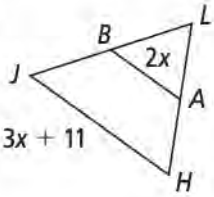
8. **5**



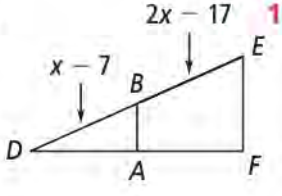
9. **1.3**



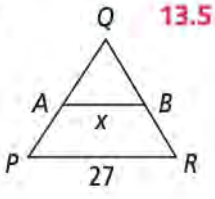
10. **11**



11. **10**



12. **13.5**



5-1 Think About a Plan

Midsegments of Triangles

Coordinate Geometry The coordinates of the vertices of a triangle are $E(1, 2)$, $F(5, 6)$, and $G(3, -2)$.

a. Find the coordinates of H, the midpoint of $\overline{EG}$, and J, the midpoint of $\overline{FG}$.

b. Show that $\overline{HJ} \parallel \overline{EF}$.

c. Show that $HJ = \frac{1}{2}EF$.

1. In part (a), what formula would you use to find the midpoints of $\overline{EG}$ and $\overline{FG}$? Write this formula. **Midpoint Formula; $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$**

2. Substitute the x- and y-coordinates of E and G into the formula. $\left(\frac{1 + 3}{2}, \frac{2 + (-2)}{2}\right)$

3. Simplify to find the coordinates of H, the midpoint of $\overline{EG}$. **(2, 0)**

4. Use the coordinates of F and G to find the coordinates of J, the midpoint of $\overline{FG}$. **(4, 2)**

5. In part (b), what information do you need to show $\overline{HJ} \parallel \overline{EF}$? Write the formula you would use. **need to show the slopes are equal; $m = \frac{y_2 - y_1}{x_2 - x_1}$**

6. Substitute the x- and y-coordinates of H and J into the formula. $m = \frac{2 - 0}{4 - 2}$

7. Simplify to find the slope of $\overline{HJ}$. **1**

8. Use the coordinates of E and F to find the slope of $\overline{EF}$. **1**

9. Is $\overline{HJ} \parallel \overline{EF}$? Explain. **Yes; both segments have the same slope.**

10. In part (c), what formula would you use to find HJ and EF? Write this formula.
Distance Formula; $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

11. Substitute the x- and y-coordinates of H and J into the formula.
$d = \sqrt{(4 - 2)^2 + (2 - 0)^2}$

12. Simplify to find HJ. Keep in simplest radical form. $2\sqrt{2}$

13. Use the coordinates of E and F to find EF. Keep in simplest radical form.
$d = \sqrt{(5 - 1)^2 + (6 - 2)^2} = 4\sqrt{2}$

14. What is the relationship between HJ and EF? $HJ = \frac{1}{2}EF$

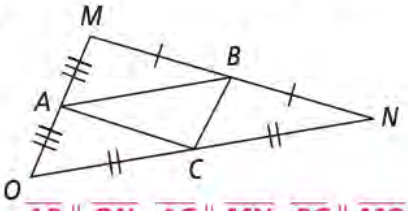
THE SCHOOL DISTRICT OF PHILADELPHIA

5-1 Practice

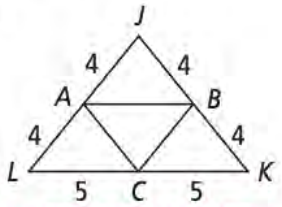
Form G

Midsegments of Triangles

Identify three pairs of triangle sides in each diagram.

1. 

$\overline{AB} \parallel \overline{ON}$; $\overline{AC} \parallel \overline{MN}$; $\overline{BC} \parallel \overline{MO}$

2. 

$\overline{AB} \parallel \overline{LK}$; $\overline{AC} \parallel \overline{JK}$; $\overline{BC} \parallel \overline{JL}$

Name the triangle sides that are parallel to the given side.

3. $\overline{AB}$ $\overline{ZY}$

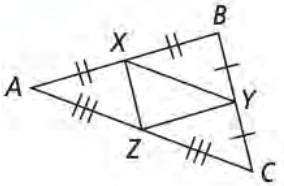
4. $\overline{AC}$ $\overline{XY}$

5. $\overline{CB}$ $\overline{ZX}$

6. $\overline{XY}$ $\overline{AC}$

7. $\overline{XZ}$ $\overline{BC}$

8. $\overline{ZY}$ $\overline{AB}$



Points *M*, *N*, and *P* are the midpoints of the sides of $\triangle QRS$. $QR = 30$, $RS = 30$, and $SQ = 18$.

9. Find *MN*. 9

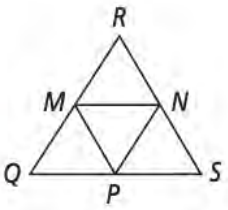
10. Find *MQ*. 15

11. Find *MP*. 15

12. Find *PS*. 9

13. Find *PN*. 15

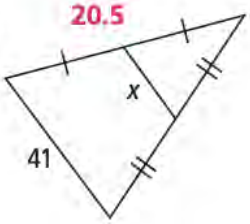
14. Find *RN*. 15



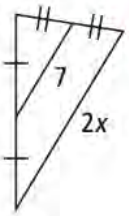
Algebra Find the value of *x*.

15. 17

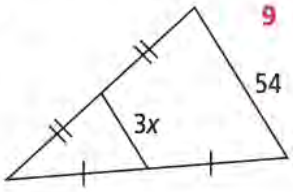
16. 20.5



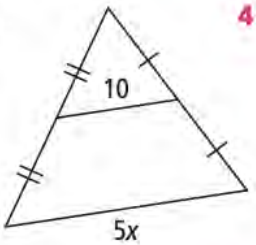
17. 7



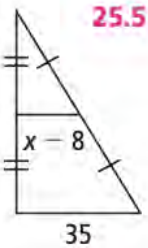
18. 9



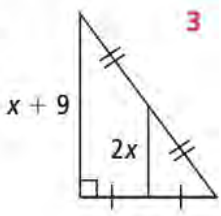
19. 4



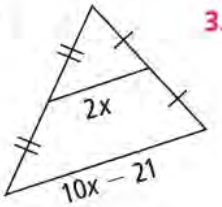
20. 25.5



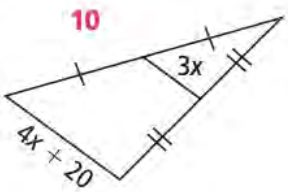
21. 3



22. 3.5



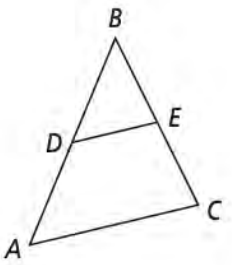
23. 10



5-1 Practice (continued) Form G

Midsegments of Triangles

***D* is the midpoint of $\overline{AB}$. *E* is the midpoint of $\overline{CB}$.**



24. If $m\angle A = 70$, find $m\angle BDE$. **70**

25. If $m\angle BED = 73$, find $m\angle C$. **73**

26. If $DE = 23$, find AC. **46**

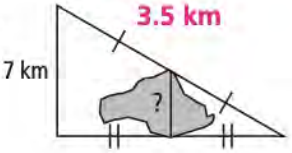
27. If $AC = 83$, find DE. **41.5**

Find the distance across the lake in each diagram.

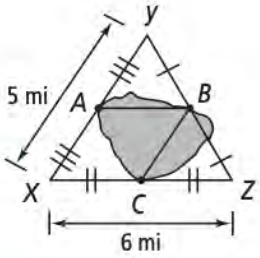
28. **13 mi**

29. **2.9 mi**

30. **3.5 km**



Use the diagram at the right for Exercises 31 and 32.



31. Which segment is shorter for kayaking across the lake, $\overline{AB}$ or $\overline{BC}$? Explain.
$\overline{BC}$ is shorter because *BC* is half of 5 mi, while *AB* is half of 6 mi.

32. Which distance is shorter, kayaking from *A* to *B* to *C*, or walking from *A* to *X* to *C*? Explain.
Neither; the distance is the same because $\overline{BC} \cong \overline{AX}$ and $\overline{AB} \cong \overline{XC}$.

33. Open-Ended Draw a triangle and all of its midsegments. Make a conjecture about what appears to be true about the four triangles that result. What postulates could be used to prove the conjecture? **Check students' drawings. Conjecture: The four triangles formed by the midsegments of a triangle are congruent. The SAS or SSS postulates can be used in each case to show that each triangle is congruent to the others.**

34. Coordinate Geometry The coordinates of the vertices of a triangle are $K(2, 3)$, $L(-2, -1)$, and $M(5, 1)$.

a. Find the coordinates of *N*, the midpoint of $\overline{KM}$, and *P*, the midpoint of $\overline{LM}$.
$N(3.5, 2)$; $P(1.5, 0)$

b. Show that $\overline{NP} \parallel \overline{KL}$. **The slope of $\overline{NP} = \frac{2-0}{3.5-1.5} = 1$ and the slope of $\overline{KL} = \frac{3-(-1)}{2-(-2)} = 1$. Because the slopes are equal, $\overline{NP} \parallel \overline{KL}$.**

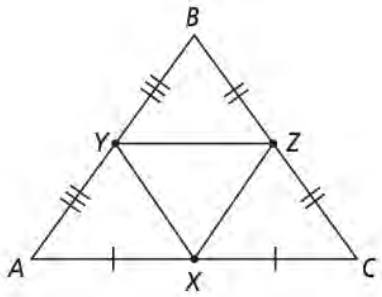
c. Show that $NP = \frac{1}{2}KL$. **$NP = \sqrt{(3.5-1.5)^2 + (2-0)^2} = 2\sqrt{2}$ and $KL = \sqrt{(-2-2)^2 + (-1-3)^2} = 4\sqrt{2}$ so $NP = \frac{1}{2}KL$.**

5-1 Practice

Form K

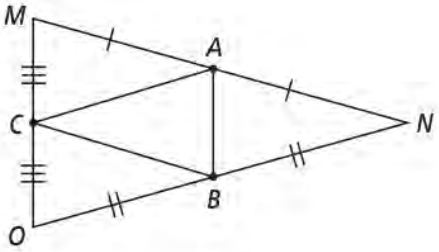
Midsegments of Triangles

Identify three pairs of parallel sides in the diagram.



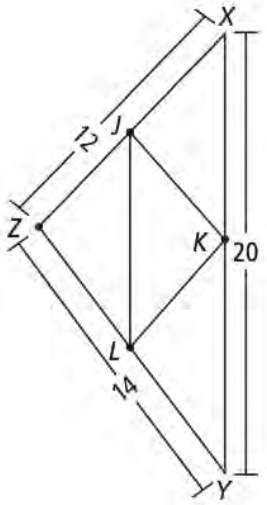
1. $\overline{AB} \parallel$? $\overline{XZ}$
2. $\overline{BC} \parallel$? $\overline{YX}$
3. $\overline{AC} \parallel$? $\overline{YZ}$

Name the side that is parallel to the given side.



4. $\overline{MN}$ $\overline{BC}$
5. $\overline{ON}$ $\overline{AC}$
6. $\overline{AB}$ $\overline{MO}$
7. $\overline{CB}$ $\overline{MN}$
8. $\overline{OM}$ $\overline{AB}$
9. $\overline{AC}$ $\overline{ON}$

Points *J*, *K*, and *L* are the midpoints of the sides of $\triangle XYZ$.



10. Find *LK*. 6

    To start, identify what kind of segment $\overline{LK}$ is. Then identify which relationship in the Triangle Midsegment Theorem will help you find the length.

    $\overline{LK}$ is a midsegment of $\boxed{\triangle XYZ}$.

    $\overline{LK}$ is parallel to $\boxed{\overline{ZX}}$.

11. Find *YK*. 10
12. Find *JK*. 7
13. Find *XK*. 10
14. Find *JL*. 10
15. Find *YL*. 7
16. Find *KL*. 6

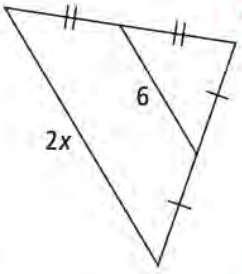
17. Draw a triangle and label it *ABC*. Draw all the midpoints and label them. Identify pairs of parallel sides and congruent angles in your triangle.
    Check students' work.

5-1 Practice (continued) Form K

Midsegments of Triangles

Algebra Find the value of x.

18. 6

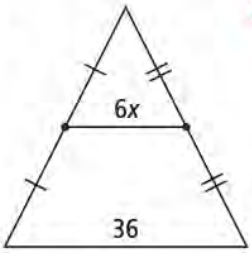


To start, identify the midsegment. Then write an equation to show that its length is half the length of its parallel segment.

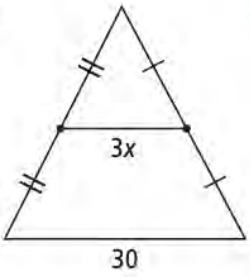
The segment with length [6] is the midsegment.

[6] $= \frac{1}{2} \cdot$ [2x]

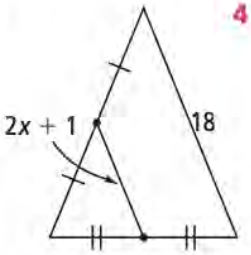
19. 3



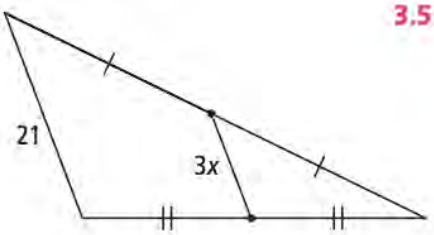
20. 5



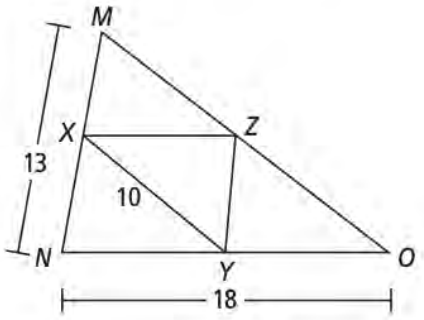
21. 4



22. 3.5



***X* is the midpoint of $\overline{MN}$. *Y* is the midpoint of $\overline{ON}$.**

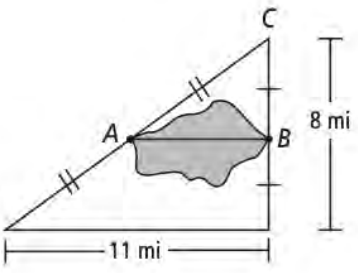


23. Find XZ. 9

24. If $XY = 10$, find MO. 20

25. If $m\angle M$ is 64, find $m\angle XYZ$. 64

Use the diagram at the right for Exercises 26 and 27.



26. What is the distance across the lake?
5.5 mi

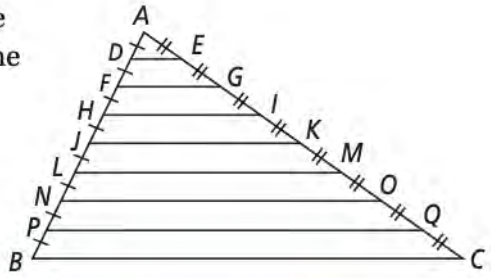
27. Is it a shorter distance from A to B or from B to C? Explain.
B to *C* is shorter; *BC* is half of 8 mi, while *AB* is half of 11 mi.

5-1 Enrichment

Midsegments of Triangles

Triangles and Maps

You can use the same reasoning behind the Triangle Midsegment Theorem to find the lengths of other line segments connecting the sides of a triangle.



Given the triangle at the right, write a paragraph proof for the following.

1. $FG = \frac{1}{4}BC$

   $JK = \frac{1}{2}BC$ by the △ Midsegment Thm. $FG = \frac{1}{2}JK$ by the △ Midsegment Thm. Therefore, $FG = \frac{1}{4}BC$.

2. $DE = \frac{1}{8}BC$

   $DE = \frac{1}{2}FG$ by the △ Midsegment Thm. $FG = \frac{1}{4}BC$ by Exercise 1. Therefore, $DE = \frac{1}{8}BC$.

3. $NO = \frac{3}{4}BC$

   $\frac{AN}{AB} = \frac{AO}{AC} = \frac{3}{4}$. By SAS ~, $\triangle ANO \sim \triangle ABC$, so $NO = \frac{3}{4}BC$.

4. $HI = \frac{3}{8}BC$

   $HI = \frac{1}{2}NO$ by the △ Midsegment Thm. $NO = \frac{3}{4}BC$ by Exercise 3. Therefore, $HI = \frac{3}{8}BC$.

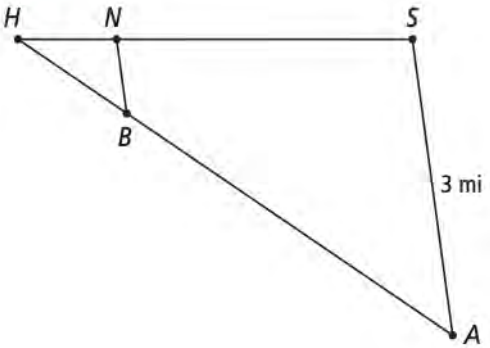
5. $LM = \frac{5}{8}BC$

   $\frac{AL}{AB} = \frac{AM}{AC} = \frac{5}{8}$. By SAS ~, $\triangle ALM \sim \triangle ABC$, so $LM = \frac{5}{8}BC$.

6. $PQ = \frac{7}{8}BC$

   $\frac{AP}{AB} = \frac{AQ}{AC} = \frac{7}{8}$. By SAS ~, $\triangle APQ \sim \triangle ABC$, so $PQ = \frac{7}{8}BC$.

Use the diagram at the right for Exercise 7.

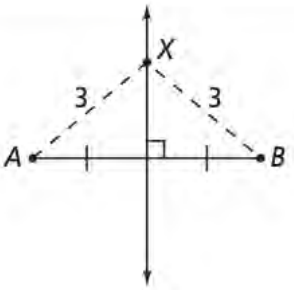
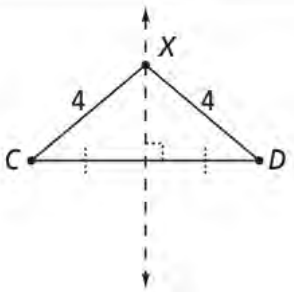
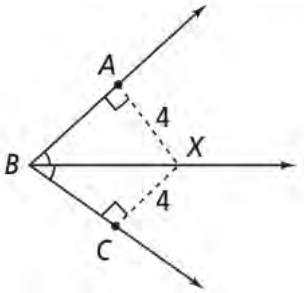
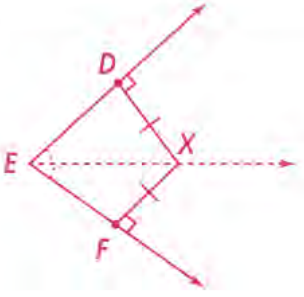


7. Nan is at point N, one-fourth of the way from her home at H to school at S. Bob is at B, which is three-quarters of the way from his apartment to Nan's home. Bob's apartment is at A, which is 3 mi from school. How far apart are Nan and Bob? 0.75 mi

5-2 Additional Vocabulary Support

Perpendicular and Angle Bisectors

Complete the vocabulary chart by filling in the missing information.

Word or Word Phrase	Statement of Theorem	Picture or Example
Perpendicular Bisector Theorem	If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.	
Converse of Perpendicular Bisector Theorem	1. If a point is equidistant from the endpoints of a segment, then it is on the perpendicular bisector of the segment.	
Angle Bisector Theorem	2. If a point is on the bisector of an angle, then the point is equidistant from the sides of the angle.	
Converse of Angle Bisector Theorem	If a point in the interior of an angle is equidistant from the sides of the angle, then the point is on the angle bisector.	3. 

5-2 Reteaching

Perpendicular and Angle Bisectors

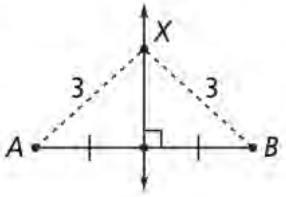
Perpendicular Bisectors

There are two useful theorems to remember about perpendicular bisectors.

Perpendicular Bisector Theorem

If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

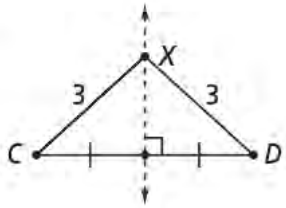
X is on the perpendicular bisector, so it is equidistant from the endpoints A and B.



Converse of the Perpendicular Bisector Theorem

If a point is equidistant from the endpoints of a segment, then it is on the perpendicular bisector of the segment.

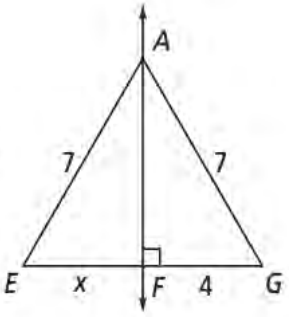
Because X is equidistant from the endpoints C and D, it is on the perpendicular bisector of the segment.



Problem

What is the value of x?

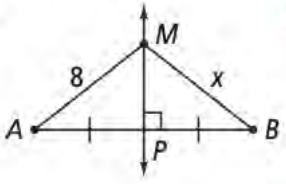
Since A is equidistant from the endpoints of the segment, it is on the perpendicular bisector of $\overline{EG}$. So, $EF = GF$ and $x = 4$.



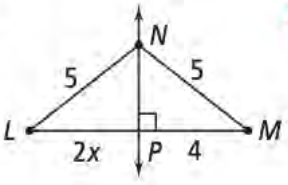
Exercises

Find the value of x.

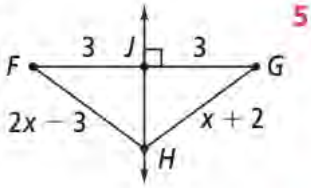
1. 8



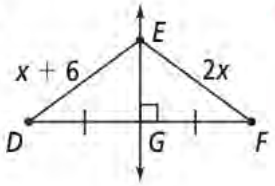
2. 2



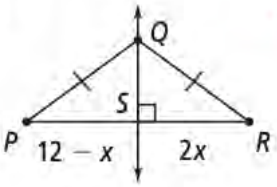
3. 5



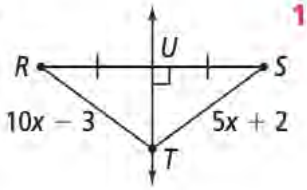
4. 6



5. 4



6. 1



5-2 Reteaching (continued)

Perpendicular and Angle Bisectors

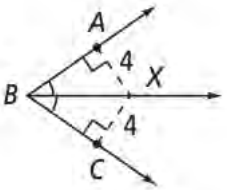
Angle Bisectors

There are two useful theorems to remember about angle bisectors.

Angle Bisector Theorem

If a point is on the bisector of an angle, then the point is equidistant from the sides of the angle.

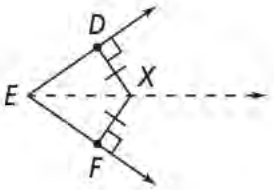
X is on the angle bisector and is therefore equidistant from the sides of the angle.



Converse of the Angle Bisector Theorem

If a point in the interior of an angle is equidistant from the sides of an angle, then the point is on the angle bisector.

Because X is in the interior of the angle and is equidistant from the sides, X is on the angle bisector.



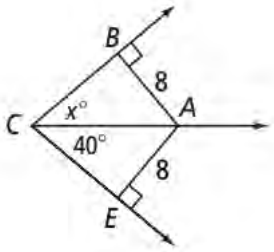
Problem

What is the value of x?

Because point A is in the interior of the angle and it is equidistant from the sides of the angle, it is on the bisector of the angle.

$$\angle BCA \cong \angle ECA$$

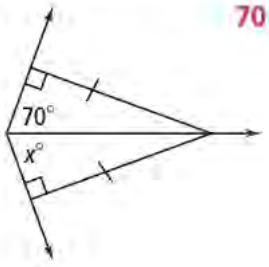
$$x = 40$$



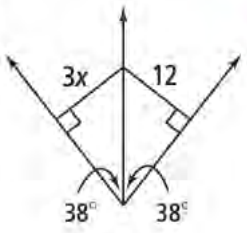
Exercises

Find the value of x.

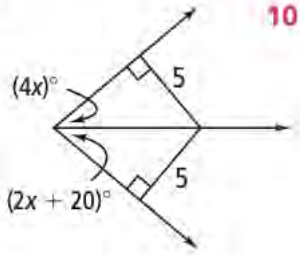
7. 70



8. 4



9. 10



5-2 Think About a Plan

Perpendicular and Angle Bisectors

a. **Constructions** Draw a large acute scalene triangle, $\triangle PQR$. Construct the perpendicular bisectors of each side.

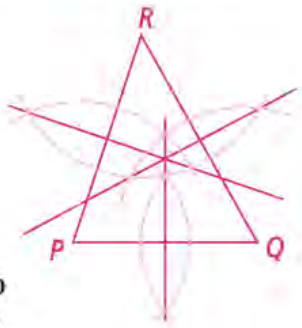
b. **Make a Conjecture** What appears to be true about the perpendicular bisectors?

c. Test your conjecture with another triangle.

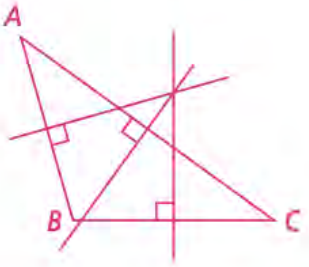
1. For part (a), what is an acute scalene triangle?

   a triangle in which all angles are acute and no side lengths are equal

2. Sketch a large acute scalene triangle. Use a protractor to make sure each angle is less than 90°. Label the vertices P, Q, and R. Check to make sure the triangle is scalene by comparing the side lengths.



3. To construct the perpendicular bisector for $\overline{PQ}$, set the compass to greater than ___$\frac{1}{2}QP$___. Draw two arcs, one from P and one from Q. The arcs ___intersect___ at two points. Draw a segment connecting the points. This segment is the ___perpendicular bisector___.



4. Construct the perpendicular bisectors of $\overline{QR}$ and $\overline{RP}$.

5. For part (b), examine the three perpendicular bisectors. Write a conjecture about the perpendicular bisectors in all triangles.

   Answers may vary. Sample: The perpendicular bisectors of any triangle intersect at one point inside the triangle.

6. For part (c), repeat Steps 1–4 for an obtuse, equilateral, or isosceles triangle. Does the conjecture appear to be true for this triangle?

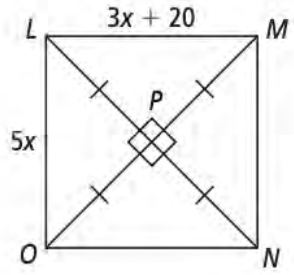
   Answers may vary. Sample: No; for an obtuse triangle, the intersection of the perpendicular bisectors lies outside the triangle.

5-2 Practice

Form G

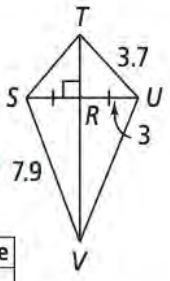
Perpendicular and Angle Bisectors

Use the figure at the right for Exercises 1–4.



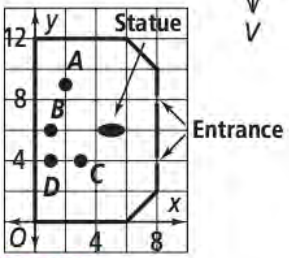
1. What is the relationship between $\overline{LN}$ and $\overline{MO}$?
   $\overline{LN}$ is the perpendicular bisector of $\overline{MO}$.
2. What is the value of x? 10
3. Find LM. 50
4. Find LO. 50

Use the figure at the right for Exercises 5–8.

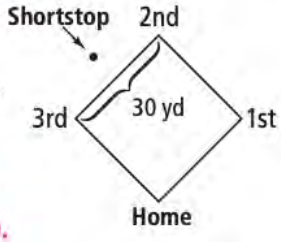


5. From the information given in the figure, how is $\overline{TV}$ related to $\overline{SU}$?
   $\overline{TV}$ is the perpendicular bisector of $\overline{SU}$.
6. Find TS. 3.7
7. Find UV. 7.9
8. Find SU. 6

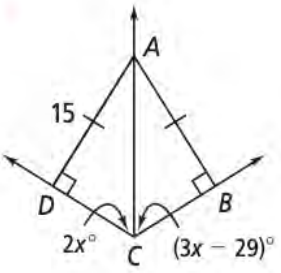
9. At the right is a layout for the lobby of a building placed on a coordinate grid.
   a. At which of the labeled points would a receptionist chair be equidistant from both entrances? B
   b. Is the statue equidistant from the entrances? How do you know? Yes; the statue is at a point that lies on the perpendicular bisector of a segment joining the entrances.



10. In baseball, the baseline is a segment connecting the bases. A shortstop is told to play back 3 yd from the baseline and exactly the same distance from second base and third base. Describe how the shortstop could estimate the correct spot. There are 30 yd between bases. Assume that the shortstop has a stride of 36 in.
    Answers may vary. Sample: Pace off 15 strides (15 yd) from third base, make a 90° left turn, and count off three more strides (3 yd).



Use the figure at the right for Exercises 11–15.

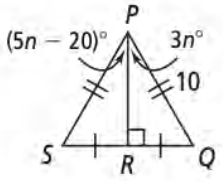


11. According to the figure, how far is A from $\overline{CD}$? From $\overline{CB}$? 15; 15
12. How is $\overrightarrow{CA}$ related to $\angle DCB$? Explain.
    $\overrightarrow{CA}$ bisects $\angle DCB$; Converse of $\angle$ Bis. Thm.
13. Find the value of x. 29
14. Find $m\angle ACD$ and $m\angle ACB$. 58; 58
15. Find $m\angle DAC$ and $m\angle BAC$. 32; 32

5-2 Practice (continued) Form G

Perpendicular and Angle Bisectors

Use the figure at the right for Exercises 16–19.



16. According to the diagram, what are the lengths of $\overline{PQ}$ and $\overline{PS}$? **10; 10**

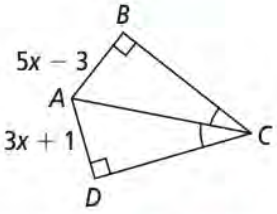
17. How is $\overline{PR}$ related to $\angle SPQ$? **$\overline{PR}$ bisects $\angle SPQ$.**

18. Find the value of n. **10**

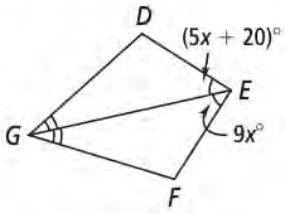
19. Find $m\angle SPR$ and $m\angle QPR$. **Both measure 30.**

Algebra Find the indicated values of the variables and measures.

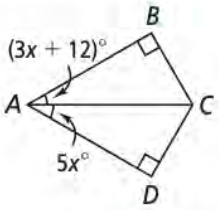
20. x, BA, DA **2; 7; 7**



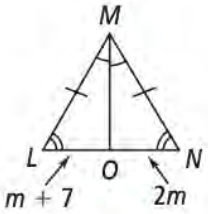
21. x, $m\angle DEF$ **5; 90**



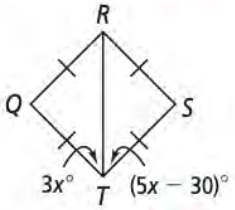
22. x, $m\angle DAB$ **6; 60**



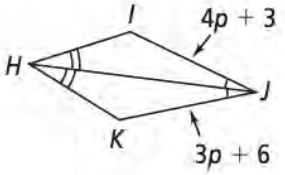
23. m, LO, NO **7; 14; 14**



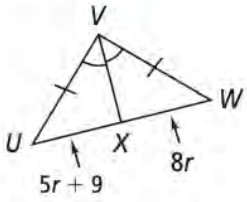
24. x, $m\angle QTS$ **15; 90**



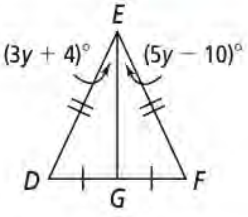
25. p, IJ, KJ **3; 15, 15**



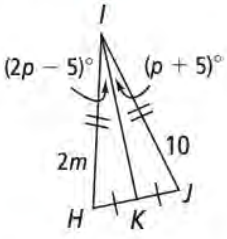
26. r, UW **3; 48**



27. y, $m\angle DEF$ **7; 50**

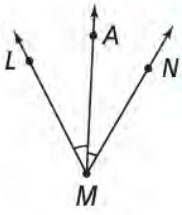


28. m, p **5; 10**



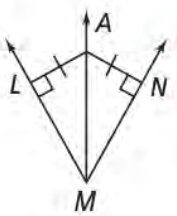
Writing Determine whether A must be on the bisector of $\angle LMN$. Explain.

29.



Yes; $\angle LMA \cong \angle NMA$, so $\overrightarrow{MA}$ is a bisector of $\angle LMN$.

30.



Yes; A is equidistant from both rays of the angle, so A lies on the bisector.

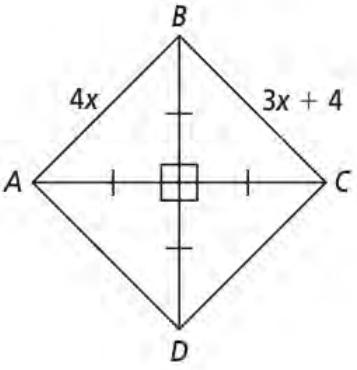
THE SCHOOL DISTRICT OF PHILADELPHIA

5-2 Practice

Form K

Perpendicular and Angle Bisectors

Use the figure at the right for Exercises 1–3.



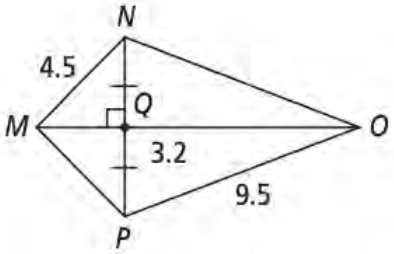
1. What is the value of x? 4

   To start, determine the relationship between $\overline{AC}$ and $\overline{BD}$. Then write an equation to show the relationships of the sides.
   $\overline{BD}$ is the ___perpendicular___ bisector of $\overline{AC}$. Therefore, point B is equidistant from points A and [C].
   $4x =$ ___ 3x + 4

2. Find AB. 16

3. Find BC. 16

Use the figure at the right for Exercises 4–7.



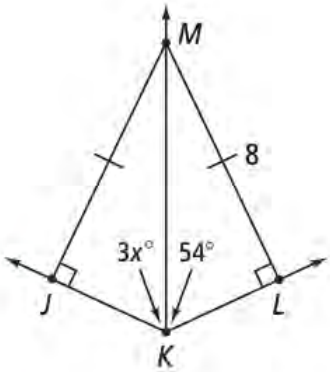
4. $\overline{MO}$ is the perpendicular bisector of [NP].

5. Find MP. 4.5

6. Find NO. 9.5

7. Find NP. 6.4

Use the figure at the right for Exercises 8–13.



8. How far is M from $\overline{KL}$? 8

9. How far is M from $\overline{JK}$? 8

10. How is $\overline{KM}$ related to $\angle JKL$? $\overline{KM}$ bisects $\angle JKL$.

11. Find the value of x. 18

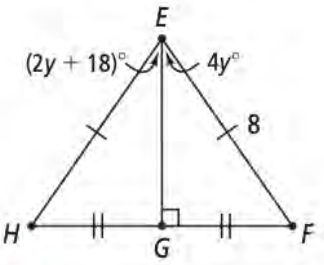
12. Find $m\angle MKL$. 54

13. Find $m\angle JMK$ and $m\angle LMK$. 36; 36

5-2 Practice (continued) Form K

Perpendicular and Angle Bisectors

Use the figure at the right for Exercises 14–16.



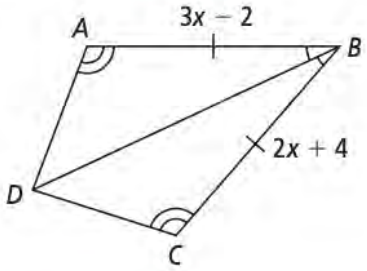
14. What are the lengths of $\overline{EF}$ and $\overline{EH}$? **8; 8**

15. Find the value of y. **9**

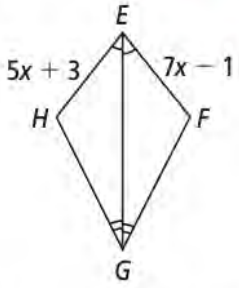
16. Find $m\angle GEH$ and $m\angle GEF$. **36; 36**

Algebra Find the indicated values of the variables and measures.

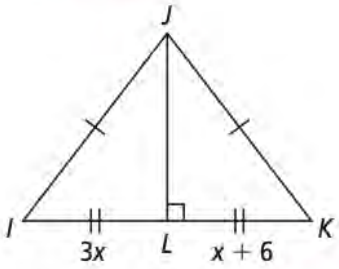
17. x, BA, BC **6; 16; 16**



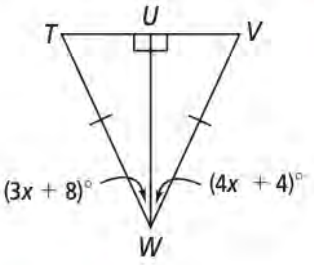
18. x, EH, EF **2; 13; 13**



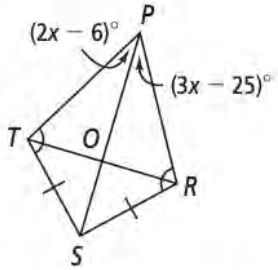
19. x, IK **3; 18**



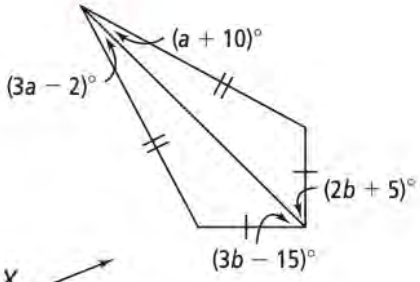
20. x, $m\angle UWV$, $m\angle UWT$ **4; 20; 20**



21. x, $m\angle TPS$, $m\angle RPS$ **19; 32; 32**

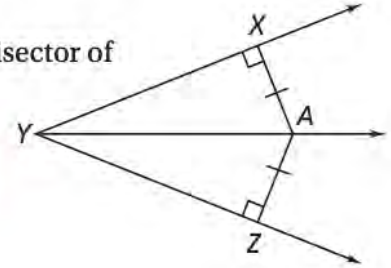


22. a, b **6; 20**



23. Writing Is A on the angle bisector of $\angle XYZ$? Explain.

Yes; A is equidistant from both rays of the angle, so A must be on the bisector.



THE SCHOOL DISTRICT OF PHILADELPHIA

5-2 Enrichment

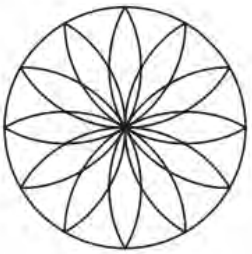
Perpendicular and Angle Bisectors

Angle Bisectors and Daisy Designs

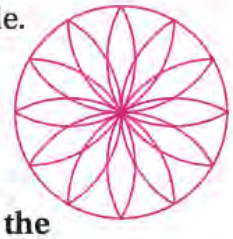
Materials

- Compass
- Straightedge

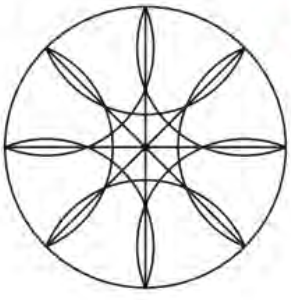
Follow the directions to replicate the daisy design.



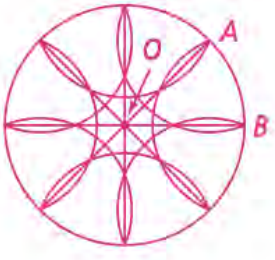
1. Use a compass to construct a circle.
2. Pick a point on the circle, keeping the radius the same, and mark a full arc that intersects the circle.
3. From the intersection point of the arc and circle, make another full arc and continue around the circle.
4. Create a point at each intersection.
5. Using two consecutive points on the circle and the center of the circle, draw an angle with the vertex at the center of the circle.
6. Bisect that angle.
7. Mark the point of intersection with the circle.
8. Using this point and the original radius of the circle, make an arc and continue around the circle. Create a point at each intersection.
9. Starting from any point, draw full arcs to connect two points on the circle.
10. Continue around the circle. When the figure is complete, erase any unnecessary marks. **Check students' work. Sample:**



Use your knowledge of straightedge and compass constructions to create the daisy design below.

11. 

Check students' work. Give hint: $m\angle AOB = 45°$. Sample:



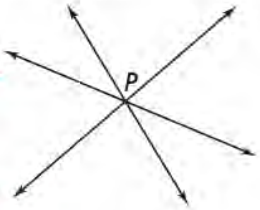
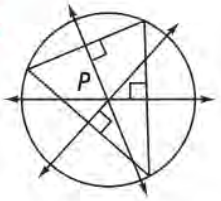
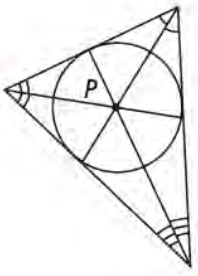
5-3 Additional Vocabulary Support

Bisectors in Triangles

For Exercises 1–5, match the term in Column A with its description in Column B. The first one is done for you.

Column A	Column B
concurrent	the point of intersection of three or more lines
1. point of concurrency	the intersection point of the three angle bisectors of a triangle
2. circumcenter of a triangle	when a circle is tangent to the three sides of a triangle
3. circumscribed about	when three or more lines intersect at a single point
4. incenter of a triangle	when a circle passes through the three vertices of a triangle
5. inscribed in	the intersection point of the three perpendicular bisectors of a triangle

For Exercises 6–8, match the phrase in Column A with the diagram in Column B that describes point *P*.

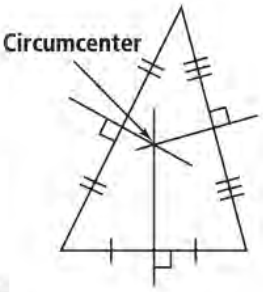
Column A	Column B
6. circumcenter of a triangle	
7. point of concurrency	
8. incenter of a triangle	

5-3 Reteaching

Bisectors in Triangles

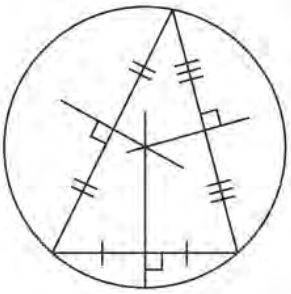
The Circumcenter of a Triangle

If you construct the perpendicular bisectors of all three sides of a triangle, the constructed segments will all intersect at one point. This point of concurrency is known as the circumcenter of the triangle.



It is important to note that the circumcenter of a triangle can lie inside, on, or outside the triangle.

The circumcenter is equidistant from the three vertices. Because of this, you can construct a circle centered on the circumcenter that passes through the triangle's vertices. This is called a *circumscribed* circle.

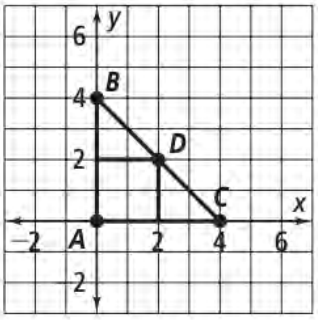


Problem

Find the circumcenter of right $\triangle ABC$.

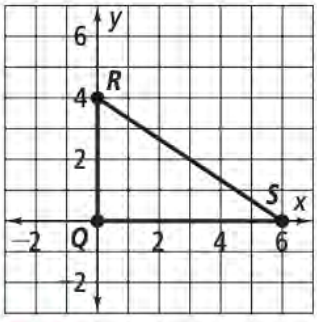
First construct perpendicular bisectors of the two legs, $\overline{AB}$ and $\overline{AC}$. These intersect at (2, 2), the circumcenter.

Notice that for a right triangle, the circumcenter is on the hypotenuse.

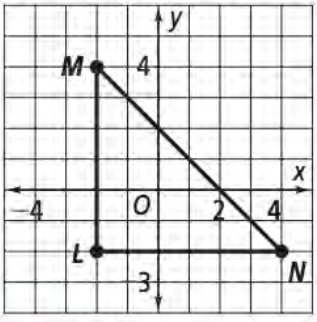


Exercises

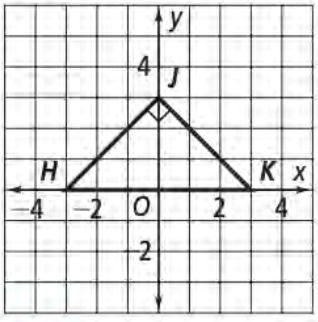
Coordinate Geometry Find the circumcenter of each right triangle.

1. 

(3, 2)

2. 

(1, 1)

3. 

(0, 0)

Coordinate Geometry Find the circumcenter of $\triangle ABC$.

4. $A(0, 0)$ (5, 4)
   $B(0, 8)$
   $C(10, 8)$

5. $A(-7, 3)$ (1, −2)
   $B(9, 3)$
   $C(-7, -7)$

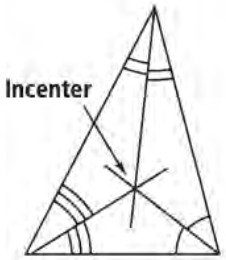
6. $A(-5, 2)$ (−1, 4)
   $B(3, 2)$
   $C(3, 6)$

5-3 Reteaching (continued)

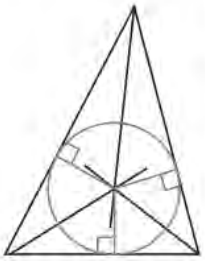
Bisectors in Triangles

The Incenter of a Triangle

If you construct angle bisectors at the three vertices of a triangle, the segments will intersect at one point. This point of concurrency where the angle bisectors intersect is known as the *incenter of the triangle.*



It is important to note that the incenter of a triangle will always lie inside the triangle.



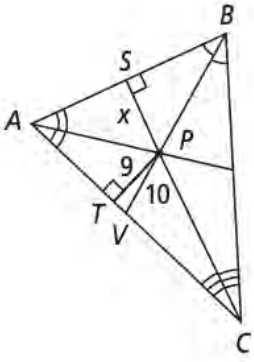
The incenter is equidistant from the sides of the triangle. You can draw a circle centered on the incenter that just touches the three sides of the triangle. This is called an *inscribed* circle.

Problem

Find the value of x.

The angle bisectors intersect at P. The incenter P is equidistant from the sides, so $SP = PT$. Therefore, $x = 9$.

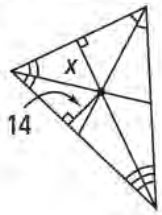
Note that $\overline{PV}$, the continuation of the angle bisector, is not the correct segment to use for the shortest distance from P to $\overline{AC}$.



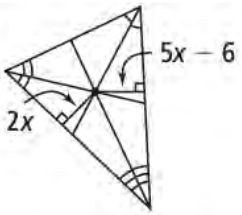
Exercises

Find the value of x.

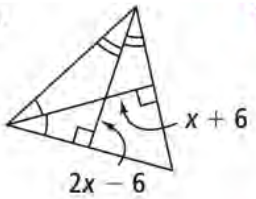
7. 14



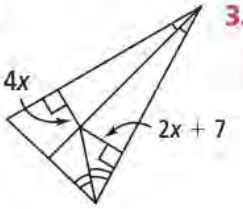
8. 2



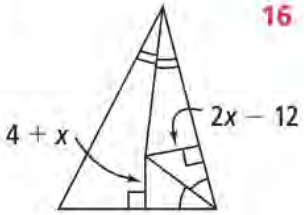
9. 12



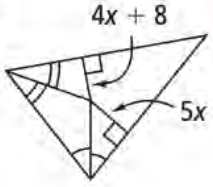
10. 3.5



11. 16



12. 8



5-3 Think About a Plan

Bisectors in Triangles

Writing Ivars found an old piece of paper inside an antique book. It read:

From the spot I buried Olaf's treasure, equal sets of paces did I measure; each of three directions in a line, there to plant a seedling Norway pine. I could not return for failing health; now the hounds of Haiti guard my wealth. —Karl

After searching Caribbean islands for five years, Ivars found an island with three tall Norway pines. How might Ivars find where Karl buried Olaf's treasure?

Know

1. Make a sketch as you answer the questions. Check students' drawings.
2. "*From the spot I buried Olaf's treasure ...*" Mark a point X on your paper.
3. "*... equal sets of paces I did measure; each of the three directions in a line ...*" This tells you to draw segments that have an endpoint at X.
   a. Explain how you know these are segments. the words "in a line"
   b. How many segments should you draw? 3
   c. What do you know about the length of the segments? They are equal.
   d. What do you know about the endpoints of the segments?
   Each segment has an endpoint at X and an endpoint at a Norway pine.
4. You do not know in which direction to draw each segment, but you can choose three directions for your sketch. Mark the locations of the trees. Draw a triangle with the trees at its vertices. What is the name of the point where X is located? circumcenter

Need

5. Look at your sketch. What do you need to find? point X, the location of the circumcenter

Plan

6. Describe how to find the treasure. The first step is done for you.

   Step 1 Find the midpoints of each side of the triangle.

   Step 2 Use the midpoints to draw perpendicular bisectors for each side.

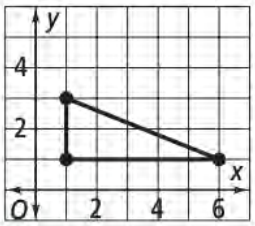
   Step 3 Find the point of concurrency of the perpendicular bisectors. This is the circumcenter, where point X is located.

5-3 Practice

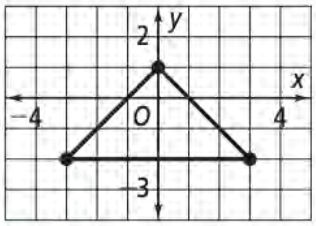
Form G

Bisectors in Triangles

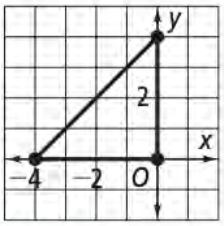
Coordinate Geometry Find the circumcenter of each triangle.

1. 

(3.5, 2)

2. 

(0, −2)

3. 

(−2, 2)

Coordinate Geometry Find the circumcenter of $\triangle ABC$.

4. $A(1, 3)$ (2.5, 2.5)
$B(4, 3)$
$C(4, 2)$

5. $A(2, -3)$ (−1, −5)
$B(-4, -3)$
$C(-4, -7)$

6. $A(-5, -2)$ (−2, 2)
$B(1, -2)$
$C(1, 6)$

7. $A(5, 6)$ (2.5, 1.5)
$B(0, 6)$
$C(0, -3)$

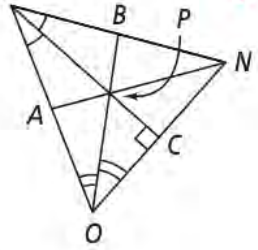
8. $A(1, 3)$ (3, 2.5)
$B(5, 3)$
$C(5, 2)$

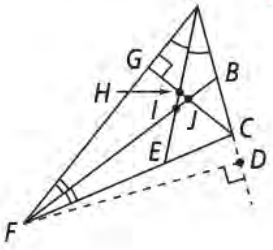
9. $A(2, -2)$ (−1, −4.5)
$B(-4, -2)$
$C(-4, -7)$

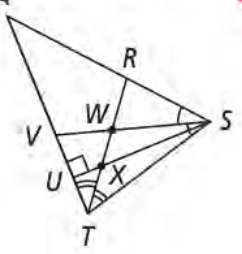
10. $A(-5, -3)$ (−2, 1.5)
$B(1, -3)$
$C(1, 6)$

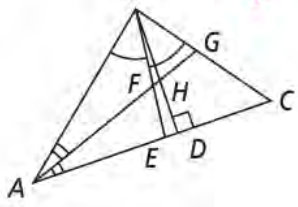
11. $A(5, 2)$ (2, −0.5)
$B(-1, 2)$
$C(-1, -3)$

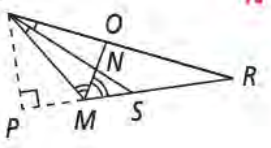
Name the point of concurrency of the angle bisectors.

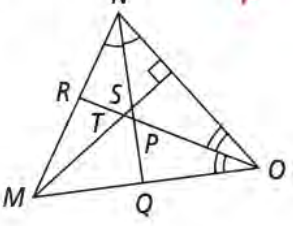
12. M P


13. A I


14. Q W


15. B F


16. Q N


17. N P


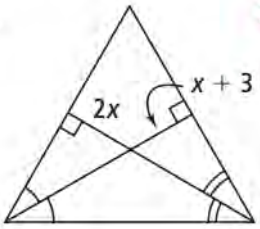
5-3 Practice (continued)

Form G

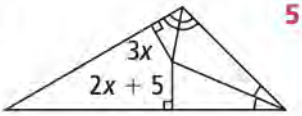
Bisectors in Triangles

Find the value of x.

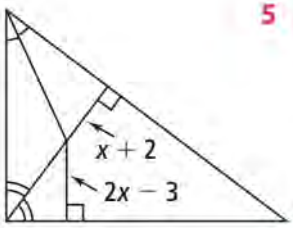
18. 3



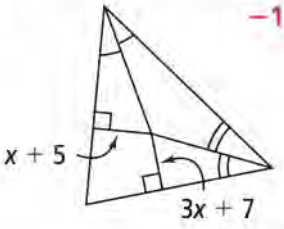
19. 5



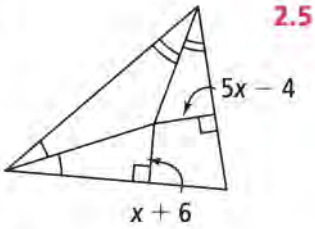
20. 5



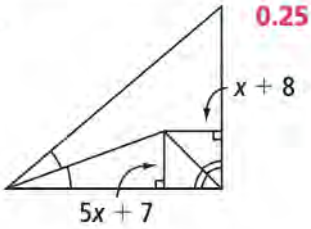
21. −1



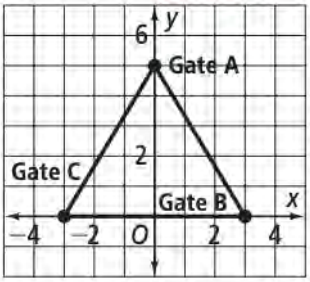
22. 2.5



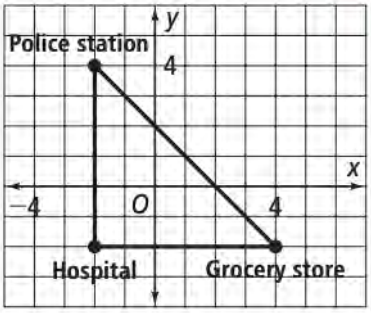
23. 0.25



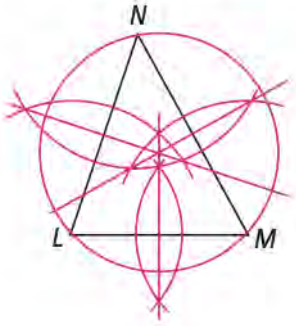
24. Where should the farmer place the hay bale so that it is equidistant from the three gates? (0, 1.6)



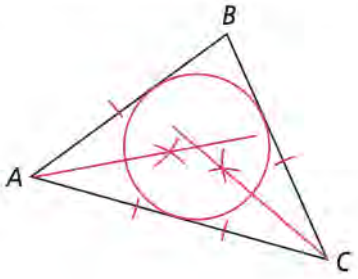
25. Where should the fire station be placed so that it is equidistant from the grocery store, the hospital, and the police station? (1, 1)



26. Construction Construct three perpendicular bisectors for $\triangle LMN$. Then use the point of concurrency to construct the circumscribed circle.



27. Construction Construct two angle bisectors for $\triangle ABC$. Then use the point of concurrency to construct the inscribed circle.

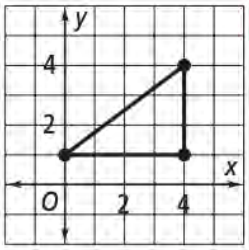


5-3 Practice

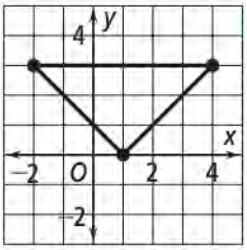
Form K

Bisectors in Triangles

Coordinate Geometry **Find the coordinates of the circumcenter of each triangle.**

1. 

(2, 2.5)

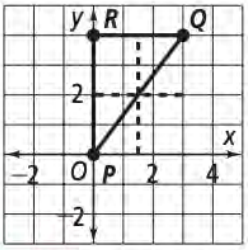
2. 

(1, 3)

Coordinate Geometry **Find the circumcenter of $\triangle PQR$.**

3. $P(0, 0)$ (1.5, 2)
$Q(3, 4)$
$R(0, 4)$

To start, graph the vertices and connect them on a coordinate plane. Then draw two perpendicular bisectors.

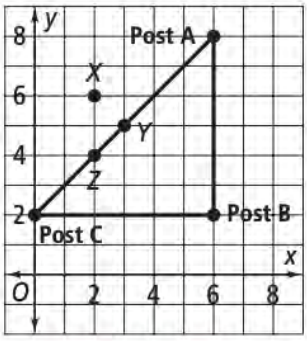


4. $P(1, -5)$ (2.5, −3.5)
$Q(4, -5)$
$R(1, -2)$

5. $P(-3, -5)$ (−1, −1.5)
$Q(-3, 2)$
$R(1, -5)$

6. $P(-6, 6)$ (−1.5, 4)
$Q(3, 6)$
$R(-6, 2)$

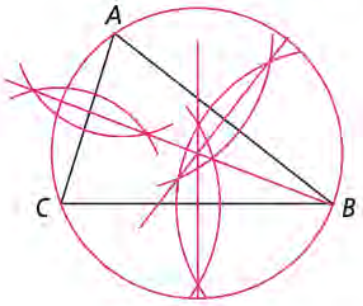
7. $P(4, 6)$ (2.5, 2)
$Q(1, 6)$
$R(1, -2)$



8. a. Which point is equidistant from the three posts? Y

b. Where are the coordinates of this point? (3, 5)

9. **Construction** Construct three perpendicular bisectors for $\triangle ABC$. Then use the point of concurrency to construct the circumscribed circle.



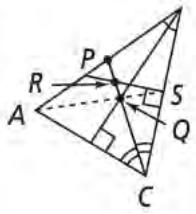
5-3 Practice (continued)

Form K

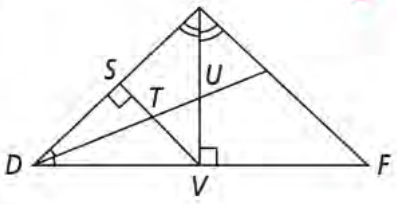
Bisectors in Triangles

Name the point of concurrency of the angle bisectors.

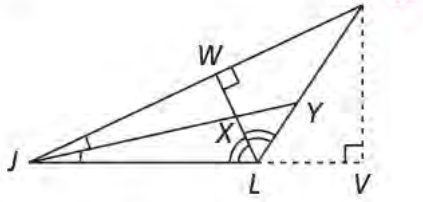
10. Q



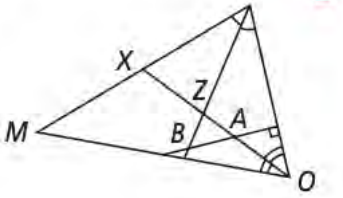
11. U



12. X



13. Z



Find the value of x.

14. To start, identify the relationship between the line segments that are labeled.

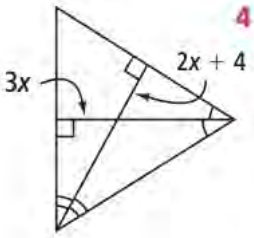
Because the segments meet at the point where the ? meet, the segments are ?.

angle bisectors; congruent

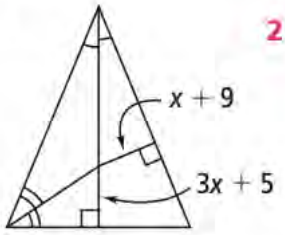
Then write an equation to find x:

$3x = 2x + 4$

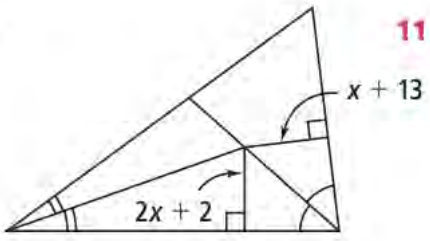
4



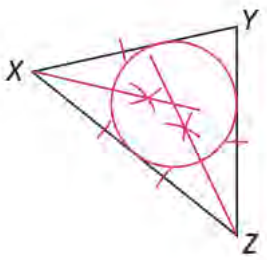
15. 2



16. 11



17. Construction Construct two angle bisectors for $\triangle XYZ$. Then use the point of concurrency to construct the inscribed circle.



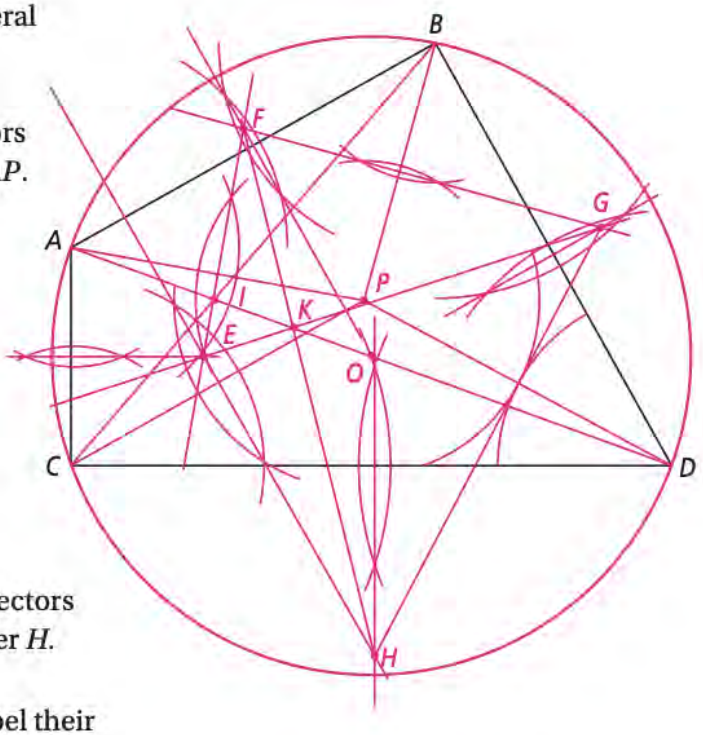
5-3 Enrichment

Bisectors in Triangles

Circumcenter of a Quadrilateral

While all triangles have a circumcenter, not all polygons have circumcenters. Quadrilaterals that have circumcenters are called *cyclic quadrilaterals,* because there is a circle that goes through all four vertices. Here you will construct the circumcenter of a given cyclic quadrilateral.

Use a compass and straightedge to perform the following construction for the circumcenter of quadrilateral *ABDC*.



1. Choose a point inside quadrilateral *ABDC*. Label it point *P*.
2. Construct perpendicular bisectors to find the circumcenter of $\triangle CAP$. Label the circumcenter *E*.
3. Construct the perpendicular bisectors of $\triangle ABP$. Label the circumcenter *F*.
4. Construct the perpendicular bisectors of $\triangle BDP$. Label the circumcenter *G*.
5. Construct the perpendicular bisectors of $\triangle DCP$. Label the circumcenter *H*.
6. Draw diagonals $\overline{AD}$ and $\overline{BC}$. Label their intersection point *I*.
7. Draw $\overleftrightarrow{EG}$ and $\overleftrightarrow{FH}$. Label their intersection point *K*.
8. Draw $\overleftrightarrow{KI}$.
9. Mark a point on $\overleftrightarrow{KI}$ that is the same distance from *K* as is *I*, but on the other side of *K*. Label this *O*. This is the circumcenter.
10. To draw a circle that circumscribes quadrilateral *ABDC*, put your compass point at *O* and your pencil on one of the vertices and draw a circle through the vertices of the quadrilateral.

5-4 Additional Vocabulary Support

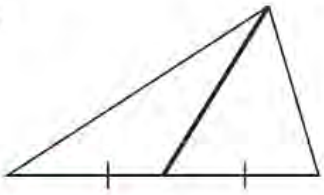
Medians and Altitudes

Concept List

altitude	centroid	circumcenter
concurrent lines	incenter	median
orthocenter	point of concurrency	vertex

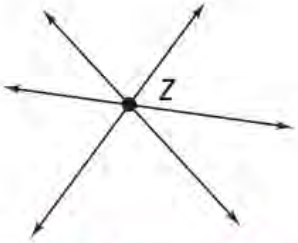
Choose the concept from the list above that best represents the item in each box.

1.



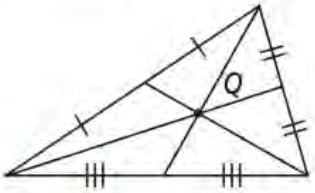
median

2. Point Z



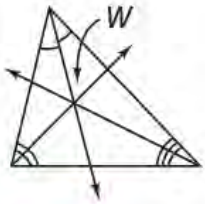
point of concurrency

3. Point Q



centroid

4. Point W

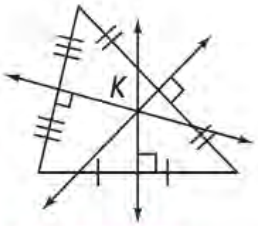


incenter

5. Three or more lines that meet at a single point

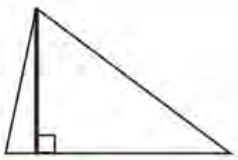
concurrent lines

6. Point K



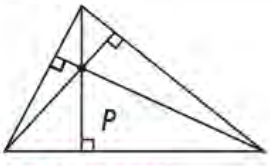
circumcenter

7.



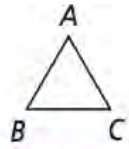
altitude

8. Point P



orthocenter

9. Point A, B, or C



vertex

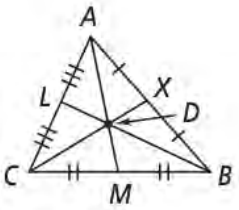
5-4 Reteaching

Medians and Altitudes

A *median* of a triangle is a segment that runs from one vertex of the triangle to the midpoint of the opposite side. The point of concurrency of the medians is called the *centroid.*

The medians of $\triangle ABC$ are $\overline{AM}$, $\overline{CX}$, and $\overline{BL}$.

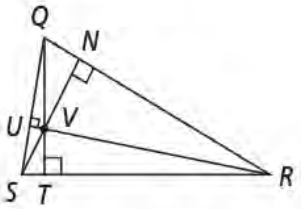
The centroid is point *D.*



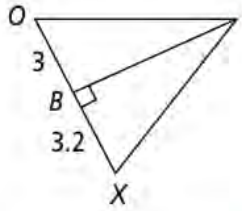
An *altitude* of a triangle is a segment that runs from one vertex perpendicular to the line that contains the opposite side. The *orthocenter* is the point of concurrency for the altitudes. An altitude may be inside or outside the triangle, or a side of the triangle.

The altitudes of $\triangle QRS$ are $\overline{QT}$, $\overline{RU}$, and $\overline{SN}$.

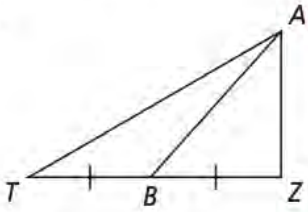
The orthocenter is point *V.*



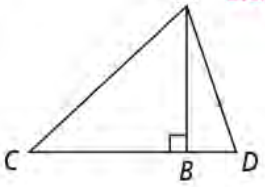
Determine whether $\overline{AB}$ is a *median,* an *altitude,* or *neither.*

1. 

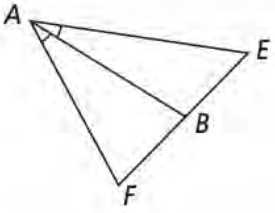
altitude

2. 

median

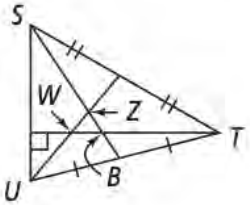
3. 

altitude

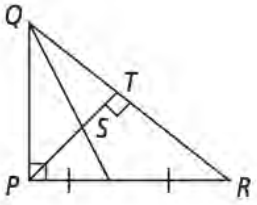
4. 

neither

5. Name the centroid. *Z*



6. Name the orthocenter. *P*



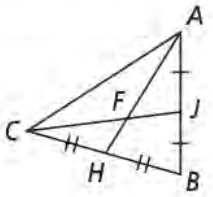
5-4 Reteaching (continued)

Medians and Altitudes

The medians of a triangle intersect at a point two-thirds of the distance from a vertex to the opposite side. This is the Concurrency of Medians Theorem.

$\overline{CJ}$ and $\overline{AH}$ are medians of $\triangle ABC$ and point F is the centroid.

$$CF = \frac{2}{3}CJ$$



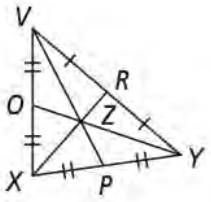
Problem

Point F is the centroid of $\triangle ABC$. If $CF = 30$, what is CJ?

$CF = \frac{2}{3}CJ$	Concurrency of Medians Theorem
$30 = \frac{2}{3} \times CJ$	Fill in known information.
$\frac{3}{2} \times 30 = CJ$	Multiply each side by $\frac{3}{2}$.
$45 = CJ$	Solve for CJ.

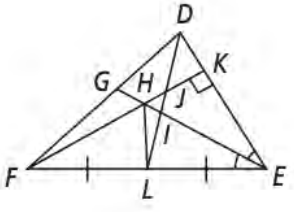
Exercises

In $\triangle VYX$, the centroid is Z. Use the diagram to solve the problems.



7. If $XR = 24$, find XZ and ZR. 16; 8
8. If $XZ = 44$, find XR and ZR. 66; 22
9. If $VZ = 14$, find VP and ZP. 21; 7
10. If $VP = 51$, find VZ and ZP. 34; 17
11. If $ZO = 10$, find YZ and YO. 20; 30
12. If $YO = 18$, find YZ and ZO. 12; 6

In Exercises 13–16, name each segment.



13. a median in $\triangle DEF$ $\overline{DL}$
14. an altitude in $\triangle DEF$ $\overline{FK}$
15. a median in $\triangle EHF$ $\overline{HL}$
16. an altitude in $\triangle HEK$ $\overline{HK}$ or $\overline{KE}$

5-4 Think About a Plan

Medians and Altitudes

Coordinate Geometry $\triangle ABC$ has vertices $A(0, 0)$, $B(2, 6)$, and $C(8, 0)$. Define the points L, M, and N such that $AL = LB$, $BM = MC$, and $CN = NA$. Complete the following steps to verify the Concurrency of Medians Theorem for $\triangle ABC$.

a. Find the coordinates of midpoints L, M, and N.

b. Find equations of $\overleftrightarrow{AM}$, $\overleftrightarrow{BN}$, and $\overleftrightarrow{CL}$.

c. Find the coordinates of P, the intersection of $\overleftrightarrow{AM}$ and $\overleftrightarrow{BN}$. This is the centroid.

d. Show that point P is on $\overleftrightarrow{CL}$.

e. Use the Distance Formula to show that point P is two-thirds of the distance from each vertex to the midpoint of the opposite side.

1. Write the midpoint formula. $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

2. Use the formula to find the coordinates of L, M, and N. (1, 3); (5, 3); (4, 0)

3. Find the slopes of $\overleftrightarrow{AM}$, $\overleftrightarrow{BN}$, and $\overleftrightarrow{CL}$. $\frac{3}{5}$; -3; $-\frac{3}{7}$

4. Write the general point-slope form of a linear equation. $y - y_1 = m(x - x_1)$

5. Write the point-slope form equations of $\overleftrightarrow{AM}$, $\overleftrightarrow{BN}$, and $\overleftrightarrow{CL}$. $y = \frac{3}{5}x$ or $y - 3 = \frac{3}{5}(x - 5)$; $y = -3(x - 4)$ or $y - 6 = -3(x - 2)$; $y = -\frac{3}{7}(x - 8)$ or $y - 3 = -\frac{3}{7}(x - 1)$

6. Solve the system of equations for $\overleftrightarrow{AM}$ and $\overleftrightarrow{BN}$ to find the point of intersection. $(3\frac{1}{3}, 2)$

7. Show that the coordinates of point P satisfy the equation of $\overleftrightarrow{CL}$. $2 = -\frac{3}{7}(3\frac{1}{3} - 8)$

8. Use the distance formula to find AM, BN, and CL. Use a calculator and round to the nearest hundredth. 5.83; 6.32; 7.62

9. Use the distance formula to find AP, BP, and CP. 3.89; 4.22; 5.08

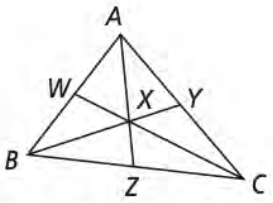
10. Check to see that $AP = \frac{2}{3}AM$, $BP = \frac{2}{3}BN$, and $CP = \frac{2}{3}CL$.
$5.83 \times \frac{2}{3} = 3.89$; $6.32 \times \frac{2}{3} = 4.21$; $7.62 \times \frac{2}{3} = 5.08$

5-4 Practice

Form G

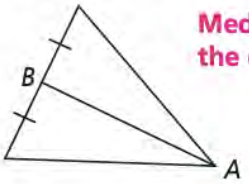
Medians and Altitudes

In $\triangle ABC$, X is the centroid.

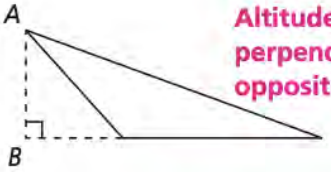


1. If $CW = 15$, find CX and XW. **$CX = 10$; $XW = 5$**
2. If $BX = 8$, find BY and XY. **$BY = 12$; $XY = 4$**
3. If $XZ = 3$, find AX and AZ. **$AX = 6$; $AZ = 9$**

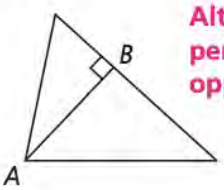
Is $\overline{AB}$ a *median*, an *altitude*, or *neither?* Explain.

4. 

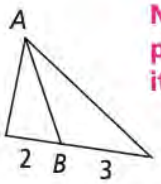
Median; $\overline{AB}$ bisects the opposite side.

5. 

Altitude; $\overline{AB}$ is perpendicular to the opposite side.

6. 

Altitude; $\overline{AB}$ is perpendicular to the opposite side.

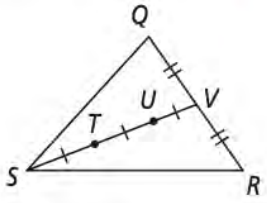
7. 

Neither; $\overline{AB}$ is not perpendicular to nor does it bisect the opposite side.

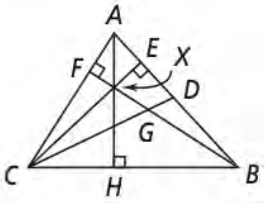
Coordinate Geometry Find the orthocenter of $\triangle ABC$.

8. $A(2, 0)$, $B(2, 4)$, $C(6, 0)$ **(2, 0)**
9. $A(1, 1)$, $B(3, 4)$, $C(6, 1)$ **(3, 3)**

10. Name the centroid. ***U***

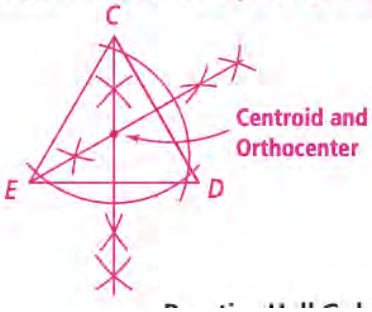


11. Name the orthocenter. ***X***



Draw a triangle that fits the given description. Then construct the centroid and the orthocenter.

12. equilateral $\triangle CDE$ **Sample: See art.**



13. acute isosceles $\triangle XYZ$ **Sample: See art.**

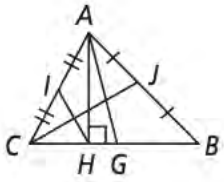
5-4 Practice (continued) Form G

Medians and Altitudes

In Exercises 14–18, name each segment.

14. a median in $\triangle ABC$ $\overline{CJ}$

15. an altitude for $\triangle ABC$ $\overline{AH}$



16. a median in $\triangle AHC$ $\overline{IH}$

17. an altitude for $\triangle AHB$ $\overline{AH}$ or $\overline{BH}$

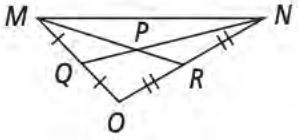
18. an altitude for $\triangle AHG$ $\overline{AH}$ or $\overline{GH}$

19. $A(0, 0)$, $B(0, -2)$, $C(-3, 0)$. Find the orthocenter of $\triangle ABC$. (0, 0)

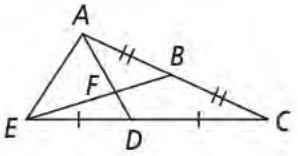
20. Cut a large isosceles triangle out of paper. Paper-fold to construct the medians and the altitudes. How are the altitude to the base and the median to the base related? They are the same.

21. In which kind of triangle is the centroid at the same point as the orthocenter? equilateral

22. P is the centroid of $\triangle MNO$. $MP = 14x + 8y$. Write expressions to represent PR and MR. $PR = 7x + 4y$; $MR = 21x + 12y$



23. F is the centroid of $\triangle ACE$. $AD = 15x^2 + 3y$. Write expressions to represent AF and FD. $AF = 10x^2 + 2y$; $FD = 5x^2 + y$



24. Use coordinate geometry to prove the following statement.

Given: $\triangle ABC$; $A(c, d)$, $B(c, e)$, $C(f, e)$

Prove: The circumcenter of $\triangle ABC$ is a point on the triangle.

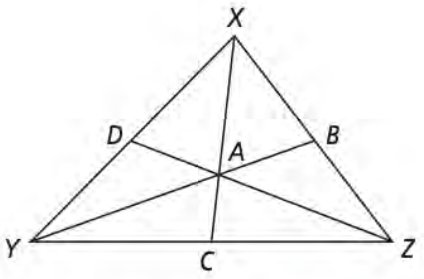
Sample: The circumcenter is the intersection of the perpendicular bisectors of a triangle. The midpoints of $\overline{AB}$ and $\overline{BC}$ are $(c, \frac{d+e}{2})$ and $(\frac{c+f}{2}, e)$. So, the equations of their perpendicular bisectors are $x = \frac{c+f}{2}$ and $y = \frac{d+e}{2}$. Their intersection is $(\frac{c+f}{2}, \frac{d+e}{2})$, which is the midpoint of AC.

5-4 Practice

Form K

Medians and Altitudes

In $\triangle XYZ$, A is the centroid.



1. If $DZ = 12$, find ZA and AD. **8; 4**

To start, write an equation relating the distance between the vertex and centroid to the length of the median.

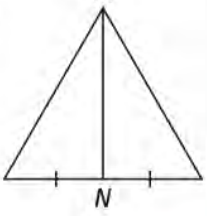
$ZA = \boxed{\frac{2}{3}} DZ$

2. If $AB = 6$, find BY and AY. **18; 12**

3. If $AC = 3$, find CX and AX. **9; 6**

Is $\overline{MN}$ a *median*, an *altitude*, or *neither*? Explain.

4.

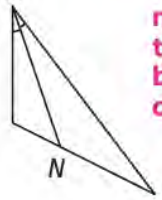


To start, identify the relationship between $\overline{MN}$ and the side it intersects.

$\overline{MN}$ __?__ the side of the triangle it intersects. **bisects**

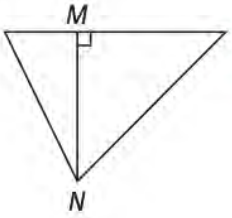
Median; $\overline{MN}$ bisects the opposite side.

5.



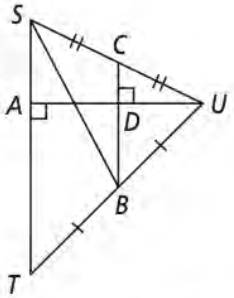
Neither; $\overline{MN}$ is not perpendicular to nor does it bisect the opposite side.

6.



Altitude; $\overline{MN}$ is perpendicular to the opposite side.

In Exercises 7–10, name each segment.



7. a median in $\triangle STU$ **$\overline{SB}$**

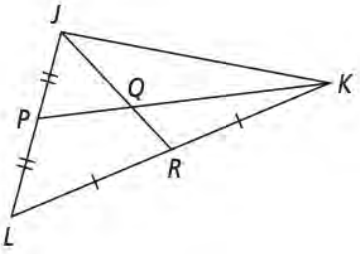
8. an altitude in $\triangle STU$ **$\overline{AU}$**

9. a median in $\triangle SBU$ **$\overline{CB}$**

10. an altitude in $\triangle CBU$ **$\overline{DU}$**

11. Q is the centroid of $\triangle JKL$. $PK = 9x + 21y$. Write expressions to represent PQ and QK.

$PQ = 3x + 7y$, $QK = 6x + 14y$

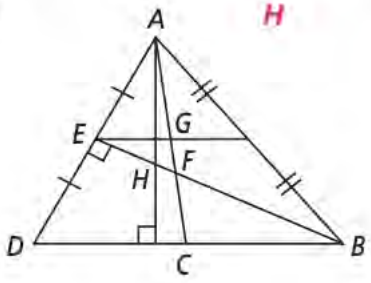


5-4 Practice (continued) Form K

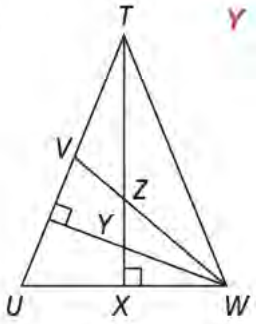
Medians and Altitudes

Find the orthocenter of each triangle.

12. H



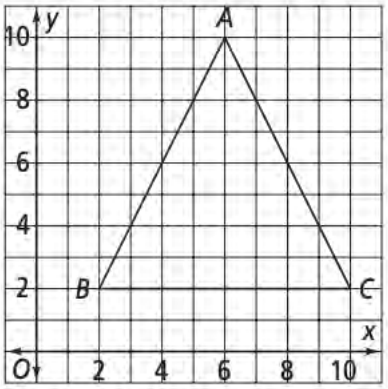
13. Y



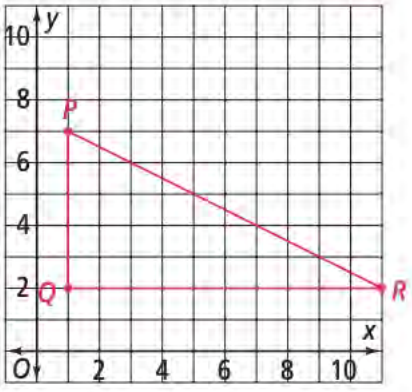
Coordinate Geometry **Find the coordinates of the orthocenter of $\triangle ABC$.**

14. $A(6, 10), B(2, 2), C(10, 2)$ **(6, 4)**

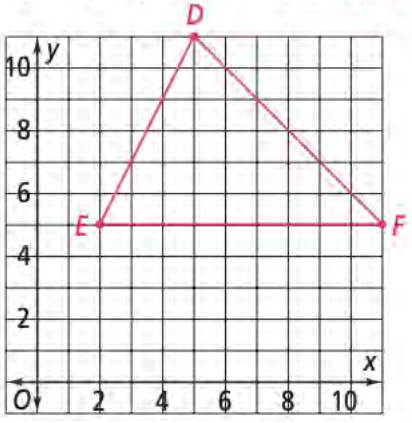
To start, graph the vertices of the triangle in a coordinate plane.



15. $P(1, 7), Q(1, 2), R(11, 2)$ **(1, 2)**

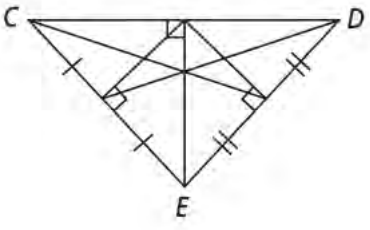


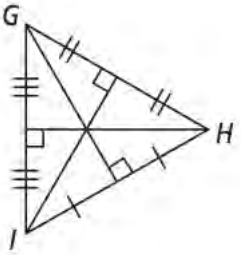
16. $D(5, 11), E(2, 5), F(11, 5)$ **(5, 8)**



17. Which triangle has a centroid at the same point as the orthocenter?

$\triangle GHI$





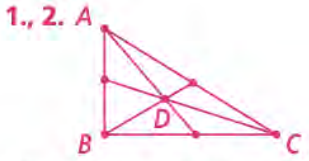
5-4 Enrichment

Medians and Altitudes

Constructions—Centroid and Orthocenter

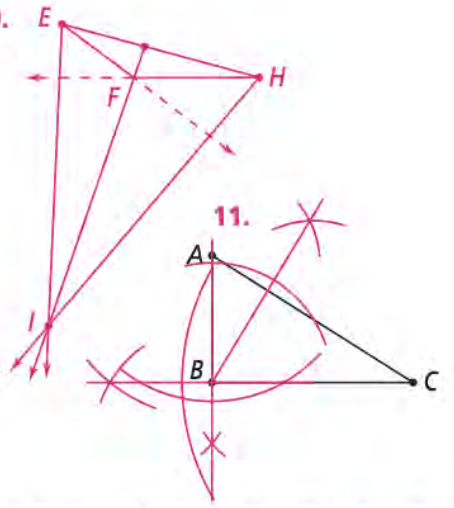
Materials

- Compass
- Straightedge
- Card stock



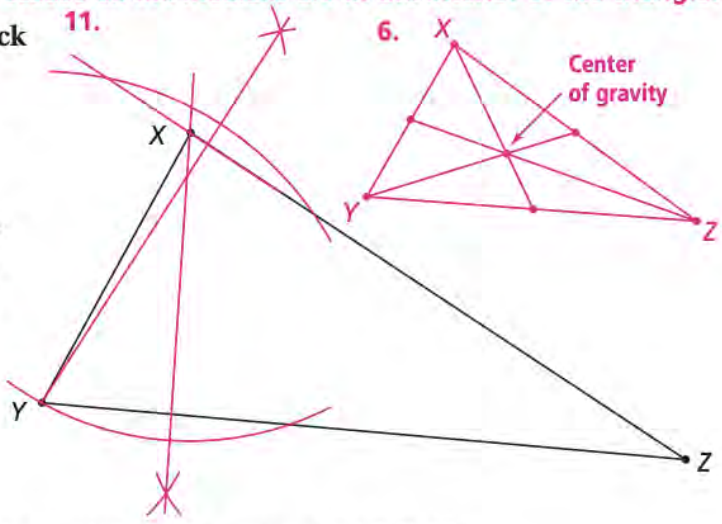
Follow the directions below to locate a point of concurrency of $\triangle ABC$.

1. Using a construction or folding, find the midpoint of each side of the triangle.
2. Draw all three medians of the triangle.
3. What word names the point of intersection of three medians? Label this point D. centroid
4. Is this point in the interior or the exterior of $\triangle ABC$? interior
5. Is it possible for this point to lie outside a triangle? Explain. It is not possible for the centroid to be on the exterior because all the medians are in the interior of the triangle.



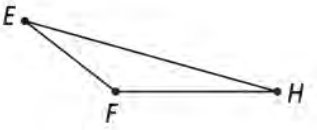
Trace $\triangle XYZ$ on a piece of card stock or cardboard, and cut it out.

6. Find the balancing point of the triangle using constructions.
7. Test your answer by placing the point on the tip of a pencil and observing whether the triangle is balanced.
   Check students' work.
8. Did you find the circumcenter, incenter, centroid, or orthocenter of the triangle?
   centroid



Follow the directions below to locate a point of concurrency of $\triangle EFH$.

9. Construct all three altitudes of $\triangle EFH$. (*Hint:* Remember that some altitudes may lie outside the triangle.)
10. What word names the point of intersection of the altitudes? Label this point I. orthocenter
11. Construct two altitudes of $\triangle ABC$ and $\triangle XYZ$. Check students' work.
12. Where do the orthocenters for obtuse, right, and acute triangles lie?
    for an obtuse triangle, exterior to the triangle; for a right triangle, on the vertex of the right angle; for an acute triangle, on the triangle interior



5-5 Additional Vocabulary Support

Indirect Proof

Use the chart below to review vocabulary. These vocabulary words will help you complete this page.

Related Words	Explanations	Examples
Assume uh SOOM	to say or believe something to be true	The weather has been cold every day this week. I *assume* it will be cold today, too.
Assumption uh SUMP shun	something taken to be true, or believed to be true	My *assumption* is that the weather in Canada is cold.
Reason (noun) REE zun	a basis for a belief or an action	My *reason* for believing that the sun will rise tomorrow is that it has always risen before.
Reason (verb) REE zun	to think or argue logically; to form conclusions	Dirk *reasons* with his sister to get her to eat her vegetables.
Reasoning REE zun ing	way of thinking, analysis	inductive *reasoning*, deductive *reasoning*
Reasonable REE zuh nuh bul	to be logical, to make sense	After solving a problem, Maria checks to see if her answer is *reasonable.*

Circle the correct answer. The first one is done for you.

I [reason/(assume)] that people use umbrellas when it rains.

1. A student multiplies 21 × 43 to get 903. She checks to see if her answer is [(reasonable)/assumption] by comparing it to the product of 20 and 40.
2. A dad could not find his children's sandals. He made the [reason/(assumption)] that his children wore the sandals to the beach.

Use the vocabulary above to fill in the blanks.

3. Before he jumped into the pool, a swimmer made the assumption that the water would be warm.
4. A girl's reason for believing that all cats purr is that every cat she knows purrs.
5. Everybody likes the chef's cooking. A diner concludes using inductive reasoning that she will enjoy the dinner the chef is cooking tonight.

5-5 Reteaching

Indirect Proof

In an *indirect proof,* you prove a statement or conclusion to be true by proving the opposite of the statement to be false.

There are three steps to writing an indirect proof.

Step 1: State as a temporary assumption the opposite (negation) of what you want to prove.

Step 2: Show that this temporary assumption leads to a contradiction.

Step 3: Conclude that the temporary assumption is false and that what you want to prove must be true.

Problem

Given: There are 13 dogs in a show; some are long-haired and the rest are short-haired. There are more long-haired than short-haired dogs.

Prove: There are at least seven long-haired dogs in the show.

Step 1: Assume that fewer than seven long-haired dogs are in the show.

Step 2: Let ℓ be the number of long-haired dogs and s be the number of short-haired dogs. Because $\ell + s = 13$, $s = 13 - \ell$. If ℓ is less than 7, s is greater than or equal to 7. Therefore, s is greater than ℓ. This *contradicts* the statement that there are more long-haired than short-haired dogs.

Step 3: Therefore, there are at least seven long-haired dogs.

Exercises

Write the temporary assumption you would make as a first step in writing an indirect proof.

1. **Given:** an integer q; **Prove:** q is a factor of 34. Assume q is not a factor of 34.
2. **Given:** $\triangle XYZ$; **Prove:** $XY + XZ > YZ$. Assume $XY + XZ \leq YZ$.
3. **Given:** rectangle $GHIJ$; **Prove:** $m\angle G = 90$ Assume $m\angle G \neq 90$.
4. **Given:** $\overline{XY}$ and $\overline{XM}$; **Prove:** $XY = XM$ Assume $XY \neq XM$.

Write a statement that contradicts the given statement.

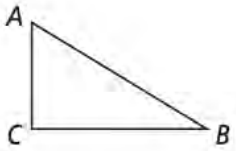
5. Whitney lives in an apartment. Whitney does not live in an apartment.
6. Marc does not have three sisters. Marc has three sisters.
7. $\angle 1$ is a right angle. $\angle 1$ is an acute angle.
8. Lines m and h intersect. Lines m and h do not intersect.

5-5 Reteaching (continued)

Indirect Proof

Problem

Given: $\angle A$ and $\angle B$ are not complementary.
Prove: $\angle C$ is not a right angle.



Step 1: Assume that $\angle C$ is a right angle.

Step 2: If $\angle C$ is a right angle, then by the Triangle Angle-Sum Theorem, $m\angle A + m\angle B + 90 = 180$. So $m\angle A + m\angle B = 90$. Therefore, $\angle A$ and $\angle B$ are complementary. But $\angle A$ and $\angle B$ are not complementary.

Step 3: Therefore, $\angle C$ is not a right angle.

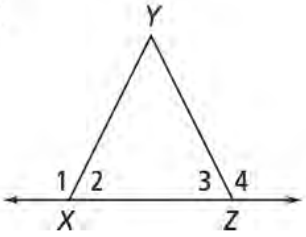
Exercises

Complete the proofs.

9. Arrange the statements given at the right to complete the steps of the indirect proof.

Given: $\overline{XY} \ncong \overline{YZ}$

Prove: $\angle 1 \ncong \angle 4$



Step 1: ? B

Step 2: ? D

Step 3: ? F

Step 4: ? E

Step 5: ? A

Step 6: ? C

A. But $\overline{XY} \ncong \overline{YZ}$.

B. Assume $\angle 1 \cong \angle 4$.

C. Therefore, $\angle 1 \ncong \angle 4$.

D. $\angle 1$ and $\angle 2$ are supplementary, and $\angle 3$ and $\angle 4$ are supplementary.

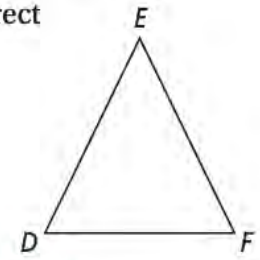
E. According to the Converse of the Isosceles Triangle Theorem, $XY = YZ$ or $\overline{XY} \cong \overline{YZ}$.

F. If $\angle 1 \cong \angle 4$, then by the Congruent Supplements Theorem, $\angle 2 \cong \angle 3$.

10. Complete the steps below to write a convincing argument using indirect reasoning.

Given: $\triangle DEF$ with $\angle D \ncong \angle F$

Prove: $\overline{EF} \ncong \overline{DE}$



Step 1: ? Assume $\overline{EF} \cong \overline{DE}$.

Step 2: ? If $\overline{EF} \cong \overline{DE}$, then by the Isosceles Triangle Theorem, $\angle D \cong \angle F$.

Step 3: ? But $\angle D \ncong \angle F$.

Step 4: ? Therefore, $\overline{EF} \ncong \overline{DE}$.

5-5 Think About a Plan

Indirect Proof

Write an indirect proof.

Given: $\triangle XYZ$ is isosceles.

Prove: Neither base angle is a right angle.

1. What is the first step in writing an indirect proof?
   State the opposite of what you want to prove as a temporary assumption.

2. Write the first step for this indirect proof.
   Assume temporarily that at least one of the base angles is a right angle.

3. What is the second step in writing an indirect proof?
   Show that the temporary assumption leads to a contradiction.

4. Find the contradiction:
   a. How are the base angle measures of an isosceles triangle related?
      The base angles have equal measures.

   b. What must be the measure of each base angle?
      90; 90

   c. What is the sum of the angle measures in a triangle? 180

   d. If both base angles of $\triangle XYZ$ are right angles, and the non-base angle has a measure greater than 0, what must be true of the sum of the angle measures?
      The sum of the angle measures must be greater than 180.

   e. What does your assumption contradict?
      the Triangle Angle-Sum Theorem

5. What is your conclusion?
   The assumption is false. Therefore, neither base angle in an isosceles triangle is a right angle.

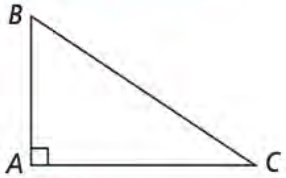
5-5 Practice

Form G

Indirect Proof

Write the first step of an indirect proof of the given statement.

1. A number g is divisible by 2.
   Assume temporarily that g is not divisible by 2.
2. There are more than three red houses on the block.
   Assume temporarily that there are at most three red houses on the block.
3. $\triangle ABC$ is equilateral.
   Assume temporarily that $\triangle ABC$ is not equilateral.
4. $m\angle B < 90$
   Assume temporarily that $m\angle B \geq 90$.
5. $\angle C$ is not a right angle.
   Assume temporarily that $\angle C$ is a right angle.
6. There are less than 15 pounds of apples in the basket.
   Assume temporarily that there are 15 or more pounds of apples in the basket.
7. If the number ends in 4, then it is not divisible by 5.
   Assume temporarily that a number that ends in 4 is divisible by 5.
8. If $\overline{MN} \cong \overline{NO}$, then point N is on the perpendicular bisector of $\overline{MO}$.
   Assume temporarily that point N is not on the perpendicular bisector of $\overline{MO}$.
9. If two right triangles have congruent hypotenuses and one pair of congruent legs, then the triangles are congruent. **Assume temporarily that there are two right triangles that are not congruent but have congruent hypotenuses and one pair of congruent legs.**
10. If two parallel lines are intersected by a transversal, then alternate interior angles are congruent. **Assume temporarily that there are two parallel lines intersected by a transversal, with alternate interior angles that are not congruent.**
11. **Developing Proof** Fill in the blanks to prove the following statement: In right $\triangle ABC$, $m\angle B + m\angle C = 90$.



Given: right $\triangle ABC$

Prove: $m\angle B + m\angle C = 90$

Assume temporarily that $m\angle B + m\angle C$ __≠90__. If $m\angle B + m\angle C$ __≠90__, then $m\angle A + m\angle B + m\angle C$ __≠180__. According to the Triangle Angle-Sum Theorem, $m\angle A + m\angle B + m\angle C =$ __180__. This contradicts the previous statement, so the temporary assumption is __false__. Therefore, __$m\angle B + m\angle C = 90$__.

12. Use indirect reasoning to eliminate all but one of the following answers.
    In what year was Barack Obama born? **C**

    (A) 1809 (B) 1909 (C) 1961 (D) 2000

5-5 Practice (continued) Form G

Indirect Proof

Identify the two statements that contradict each other.

13. I. $\triangle ABC$ is acute. II. $\triangle ABC$ is scalene. III. $\triangle ABC$ is equilateral. **II and III**

14. I. $m\angle B \leq 90$ II. $\angle B$ is acute. III. $\angle B$ is a right angle. **II and III**

15. I. $\overline{FA} \parallel \overline{AC}$
II. $\overline{FA}$ and $\overline{AC}$ are skew.
III. $\overline{FA}$ and $\overline{AC}$ do not intersect. **I and II**

16. I. Victoria has art class from 9:00 to 10:00 on Mondays. **I and III**
II. Victoria has math class from 10:30 to 11:30 on Mondays.
III. Victoria has math class from 9:00 to 10:00 on Mondays.

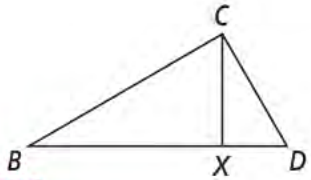
17. I. $\triangle MNO$ is acute. **II and III**
II. The centroid and the orthocenter for $\triangle MNO$ are at different points.
III. $\triangle MNO$ is equilateral.

18. I. $\triangle ABC$ such that $\angle A$ is obtuse. **I and II**
II. $\triangle ABC$ such that $\angle B$ is obtuse.
III. $\triangle ABC$ such that $\angle C$ is acute.

19. I. The orthocenter for $\triangle ABC$ is outside the triangle. **I and III**
II. The median for $\triangle ABC$ is inside the triangle.
III. $\triangle ABC$ is an acute triangle.

Write an indirect proof.

20. **Given:** $m\angle XCD = 30$, $m\angle BCX = 60$, $\angle XCD \cong \angle XBC$



Prove: $\overline{CX} \perp \overline{BD}$

Assume temporarily that $\overline{CX}$ and $\overline{BD}$ are not perpendicular. Then, $m\angle BXC \neq 90$. $m\angle BCX + m\angle BXC + m\angle XBC = 60 + m\angle BXC + 30 = 90 + m\angle BXC$. If $m\angle BXC \neq 90$, then $m\angle BCX + m\angle BXC + m\angle XBC \neq 180$. The Triangle Angle-Sum Theorem yields $m\angle BCX + m\angle BXC + m\angle XBC = 180$. This contradicts $m\angle BCX + m\angle BXC + m\angle XBC \neq 180$. So, $m\angle BXC = 90$ and $\overline{CX} \perp \overline{BD}$.

21. It is raining outside. Show that the temperature must be greater than 32°F.
Answers may vary. Sample: Suppose that the temperature is less than or equal to 32°F and it is raining. Because 32°F is the freezing point of water, any precipitation will be in the form of sleet, snow, or freezing rain. This contradicts the original statement. Therefore, the temperature must be above 32°F for it to rain.

5-5 Practice

Form K

Indirect Proof

Complete the first step of an indirect proof of the given statement.

1. There are fewer than 11 pencils in the box.
   Assume temporarily that there are __?__ pencils in the box. **11 or more**

2. If a number ends in 0, then it is not divisible by 3.
   Assume temporarily that a number that ends in 0 __?__. **is divisible by 3**

3. $4x + 3 > 12$
   Assume temporarily that $4x + 3$ $\boxed{\leq}$ 12.

4. $\triangle RST$ is not an isosceles triangle.
   Assume temporarily that __?__. **$\triangle RST$ is an isosceles triangle**

Write the first step of an indirect proof of the given statement.

5. There are more than 20 apples in a box.
   Assume temporarily that there are 20 or fewer apples in a box.

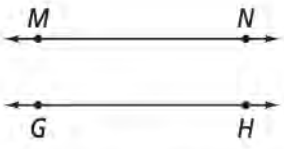
6. If a number ends in x, then it is a multiple of 5.
   Assume temporarily that a number that ends in x is not a multiple of 5.

7. $m\angle XYZ < 100$
   Assume temporarily that $m\angle XYZ \geq 100$.

8. $\triangle DEF$ is a right triangle.
   Assume temporarily that $\triangle DEF$ is not a right triangle.

Identify the two statements that contradict each other.

9. I. $\overleftrightarrow{MN} \parallel \overleftrightarrow{GH}$
   II. $\overleftrightarrow{MN}$ and $\overleftrightarrow{GH}$ do not intersect.
   III. $\overleftrightarrow{MN}$ and $\overleftrightarrow{GH}$ are skew. **I and III**



To start, identify two conditions that cannot be true at the same time.

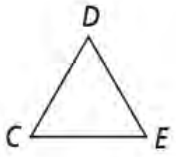
__?__ lines must be in the same plane. **Parallel**
__?__ lines must not be in the same plane. **Skew**
Therefore, two lines cannot be both __?__ and __?__. **parallel; skew**

5-5 Practice (continued) Form K

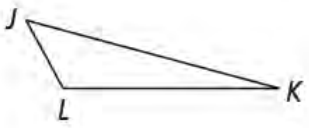
Indirect Proof

Identify the two statements that contradict each other.

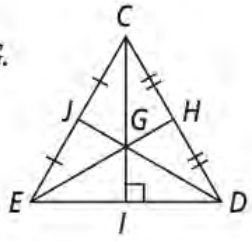
10. I. $\triangle CDE$ is equilateral.
II. $\angle C$ and $\angle E$ have the same measure.
III. $m\angle C > 60$ **I and III**



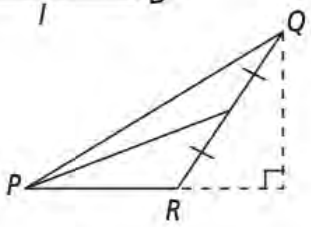
11. I. $\triangle JKL$ is scalene.
II. $\triangle JKL$ is obtuse.
III. $\triangle JKL$ is isosceles. **I and III**



12. I. The orthocenter of $\triangle CDE$ is point G.
II. The centroid and orthocenter of $\triangle CDE$ are both point G.
III. $\triangle CDE$ is scalene. **II and III**



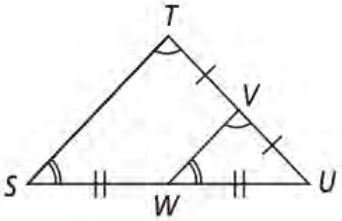
13. I. An altitude of $\triangle PQR$ is outside the triangle.
II. $\triangle PQR$ is acute.
III. An median of $\triangle PQR$ is inside the triangle. **I and II**



Complete the indirect proof.

14. Given: $\angle S \cong \angle VWU$
$\angle T \cong \angle WVU$

Prove: $\overline{TS} \parallel \overline{VW}$



Assume temporarily that ___?___. ***TS* is not parallel to *VW***

Then by the Converse of the ___?___, $\angle S$ and $\angle VWU$ cannot be ___?___. **congruent**
Corresp. ∡ Thm.

This contradicts the given information that ___?___. ***∠S ≅ ∠VWU***

Therefore, $\overline{TS}$ must be ___?___ $\overline{VW}$. **is parallel to**

5-5 Enrichment

Indirect Proof

Proofs About Triangles

Indirect reasoning is a useful way to prove things. Everyone uses indirect reasoning, sometimes without realizing it.

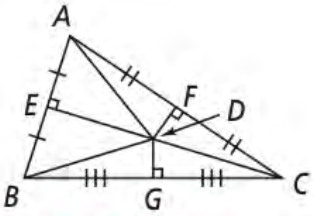
For instance, on the way to play baseball you feel a drop of water land on you, even though you had thought it was not raining. Now, however, you decide that it must be raining, even though the daylight seems too bright for such weather. You look up at the sky and see that there are no clouds, leading you to reason that it is not raining after all, and you will be able to play baseball.

You have just used the three steps of indirect reasoning. First you assume the opposite of what you want to prove. Next, you show that your assumption leads to a contradiction. Last of all, you conclude that your assumption must have been false and what you wanted to prove is true.

Use indirect reasoning to complete the following proofs of statements that have been proven using direct reasoning earlier in the chapter.

1. **Given:** $\triangle ABC$, with perpendicular bisectors meeting at point D

   Prove: $AD = BD$



   Answers may vary. Sample: Assume $AD \neq BD$. Therefore, $\overline{DE}$ is not a perpendicular bisector, because by the Perpendicular Bisector Theorem, all points on the bisector are equidistant from the endpoints of the segment. But it is given that D is the intersection of the perpendicular bisectors, so $\overline{DE}$ must be a perpendicular bisector because it is perpendicular to $\overline{AB}$ and passes through D. Therefore, $AD = BD$.

2. **Given:** The incenter and circumcenter of $\triangle JKL$ are different points.

   Prove: $\triangle JKL$ is not equilateral.

   Answers may vary. Sample: Assume $\triangle JKL$ is equilateral. The incenter of a triangle is the intersection of the angle bisectors of a triangle. The circumcenter of a triangle is the intersection of the perpendicular bisectors of a triangle. If $\triangle JKL$ is equilateral, then the perpendicular bisector $\overline{JM}$ makes two triangles, $\triangle JKM$ and $\triangle JLM$. $\overline{JM} \cong \overline{JM}$ by the Reflexive Property of $\cong$. $\overline{JK} \cong \overline{JL}$ because $\triangle JKL$ is equilateral. $\overline{MK} \cong \overline{ML}$ by the Perpendicular Bisector Theorem. So $\triangle JKM \cong \triangle JLM$ by SSS. So by CPCTC, $\angle KJM \cong \angle LJM$. Thus, $\overline{JM}$ is an angle bisector, as are the other perpendicular bisectors. So the incenter and the circumcenter are the same point. But it is given that they are different points. Therefore, $\triangle JKL$ is not equilateral.

5-6 Additional Vocabulary Support

Inequalities in One Triangle

The column on the left shows the steps used to solve an inequality using the Triangle Inequality Theorem. Use the column on the left to answer each question in the column on the right.

Problem **Triangle Inequality Theorem** **A handyman wants to make a fence around a garden in the shape of a triangle. He plans to use a 6-ft-long piece of fencing and a 7-ft-long piece of fencing. How long could the third piece of fencing be?**	**1.** Read the example. What do you need to find to solve the problem? the possible lengths of the third piece of fencing
Choose a variable to represent the length of the third piece of fencing: x.	**2.** What is a *variable?* a symbol or letter that represents an unknown number **3.** What does *represent* mean? to act as a substitute for
Set up three inequalities using the Triangle Inequality Theorem: $6 + 7 > x$ $x + 6 > 7$ $x + 7 > 6$	**4.** What is the Triangle Inequality Theorem? The sum of the lengths of any two sides of a triangle is greater than the length of the third side.
Solve the inequalities: $13 > x$ $x > 1$ $x > -1$	**5.** What does it mean to *solve an inequality?* to find the values of the variable that make the inequality true
Form a conclusion: The value of x must be less than 13 and greater than 1. So, the length of the third piece of fencing must be greater than 1 ft and less than 13 ft.	**6.** Why is there more than one possible value of x? Many different lengths of fencing could be used to form the third side of the triangle.

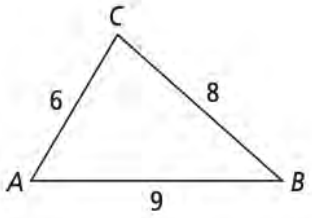
5-6 Reteaching

Inequalities in One Triangle

For any triangle, if two sides are not congruent, then the larger angle is opposite the longer side (Theorem 5-10). Conversely, if two angles are not congruent, then the longer side is opposite the larger angle (Theorem 5-11).

Problem

Use the triangle inequality theorems to answer the questions.



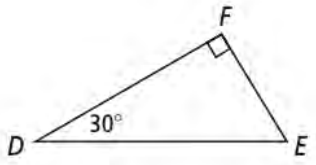
a. Which is the largest angle of $\triangle ABC$?

$\overline{AB}$ is the longest side of $\triangle ABC$. $\angle C$ lies opposite $\overline{AB}$.

$\angle C$ is the largest angle of $\triangle ABC$.

b. What is $m\angle E$? Which is the shortest side of $\triangle DEF$?

$$m\angle D + m\angle E + m\angle F = 180 \quad \text{Triangle Angle-Sum Theorem}$$
$$30 + m\angle E + 90 = 180 \quad \text{Substitution}$$
$$120 + m\angle E = 180 \quad \text{Addition}$$
$$m\angle E = 60 \quad \text{Subtraction Property of Equality}$$



$\angle D$ is the smallest angle of $\triangle DEF$. Because $\overline{FE}$ lies opposite $\angle D$, $\overline{FE}$ is the shortest side of $\triangle DEF$.

Exercises

1. Draw three triangles, one obtuse, one acute, and one right. Label the vertices. Exchange your triangles with a partner.

a. Identify the longest and shortest sides of each triangle.

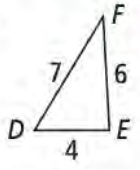
b. Identify the largest and smallest angles of each triangle.

c. Describe the relationship between the longest and shortest sides and the largest and smallest angles for each of your partner's triangles.

Check students' work. The longest side will be opposite the largest angle. The shortest side will be opposite the smallest angle.

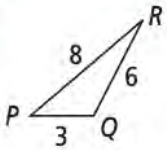
Which are the largest and smallest angles of each triangle?

2.



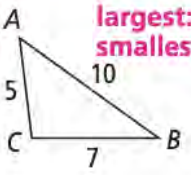
largest: $\angle DEF$; smallest: $\angle DFE$

3.



largest: $\angle PQR$; smallest: $\angle PRQ$

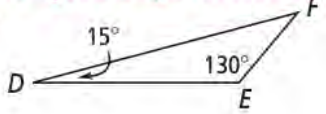
4.



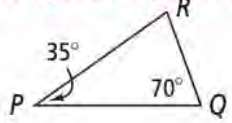
largest: $\angle ACB$; smallest: $\angle CBA$

Which are the longest and shortest sides of each triangle?

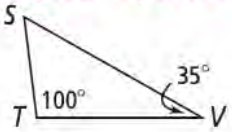
5. longest: $\overline{DF}$; shortest: $\overline{FE}$



6. longest: $\overline{PQ}$; shortest: $\overline{RQ}$



7. longest: $\overline{SV}$; shortest: $\overline{ST}$



5-6 Reteaching (continued)

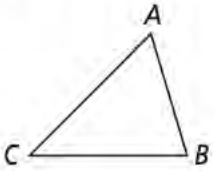
Inequalities in One Triangle

For any triangle, the sum of the lengths of any two sides is greater than the length of the third side. This is the *Triangle Inequality Theorem.*

$AB + BC > AC$

$AC + BC > AB$

$AB + AC > BC$



Problem

A. Can a triangle have side lengths 22, 33, and 25?

Compare the sum of two side lengths with the third side length.

$22 + 33 > 25$ $\quad$ $22 + 25 > 33$ $\quad$ $25 + 33 > 22$

A triangle *can* have these side lengths.

B. Can a triangle have side lengths 3, 7, and 11?

Compare the sum of two side lengths with the third side length.

$3 + 7 < 11$ $\quad$ $3 + 11 > 7$ $\quad$ $11 + 7 > 3$

A triangle *cannot* have these side lengths.

C. Two sides of a triangle are 11 and 12 ft long. What could be the length of the third side?

Set up inequalities using x to represent the length of the third side.

$x + 11 > 12$ $\quad$ $x + 12 > 11$ $\quad$ $11 + 12 > x$

$x > 1$ $\quad$ $x > -1$ $\quad$ $23 > x$

The side length can be any value between 1 and 23 ft long.

Exercises

8. Can a triangle have side lengths 2, 3, and 7? no

9. Can a triangle have side lengths 12, 13, and 7? yes

10. Can a triangle have side lengths 6, 8, and 9? yes

11. Two sides of a triangle are 5 cm and 3 cm. What could be the length of the third side? less than 8 cm and greater than 2 cm

12. Two sides of a triangle are 15 ft and 12 ft. What could be the length of the third side? less than 27 ft and greater than 3 ft

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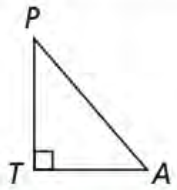
5-6 Think About a Plan

Inequalities in One Triangle

Prove this corollary to Theorem 5-11: The perpendicular segment from a point to a line is the shortest segment from the point to the line.

Given: $\overline{PT} \perp \overline{TA}$

Prove: $PA > PT$



1. What is $m\angle T$? Explain how you know this.

   90; two perpendicular line segments form right angles, and right angles are 90°.

2. What is $m\angle P + m\angle A + m\angle T$? Explain how you know this.

   180; the Triangle Angle-Sum Theorem

3. What is $m\angle P + m\angle A$? Explain how you know this.

   90; I subtracted the value of $m\angle T$ from both sides of the equation $m\angle P + m\angle A + m\angle T = 180$.

4. Write an inequality to show $m\angle A$. $0 < m\angle A < 90$

5. Write an inequality to show the relationship between $m\angle A$ and $m\angle T$. $m\angle A < m\angle T$

6. Which side lies opposite $\angle A$ and which side lies opposite $\angle T$?

   $\overline{PA}$ lies opposite $\angle T$, and $\overline{PT}$ lies opposite $\angle A$.

7. What is Theorem 5-11?

   If two angles of a triangle are not congruent, then the longer side lies opposite the larger angle.

8. What can you conclude about PA and PT?

   $PA > PT$

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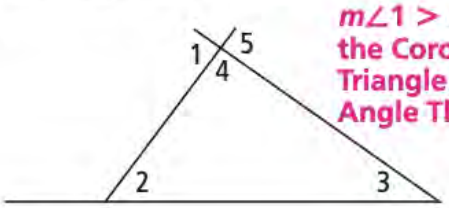
5-6 Practice

Form G

Inequalities in One Triangle

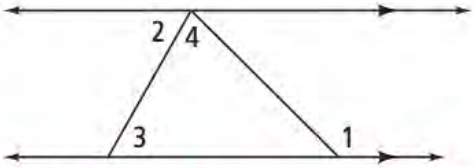
Explain why $m\angle 1 > m\angle 2$.

1.



$m\angle 1 > m\angle 2$ by the Corollary to the Triangle Exterior Angle Theorem.

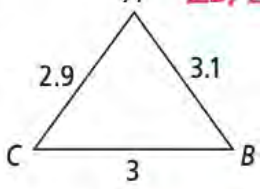
2.



$m\angle 2 = m\angle 3$ because they are Alt. Int. $\angle$s between $\parallel$ lines. By the Corollary to the $\triangle$ Ext. $\angle$ Thm., $m\angle 1 > m\angle 3$. So, $m\angle 1 > m\angle 2$ by the Trans. Prop. of Ineq.

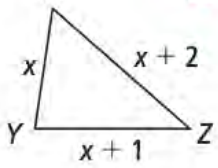
For Exercises 3–6, list the angles of each triangle in order from smallest to largest.

3.



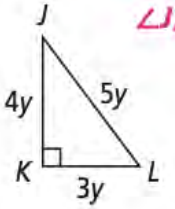
$\angle B, \angle A, \angle C$

4.



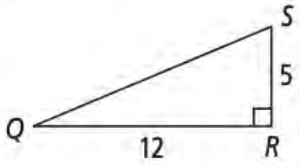
$\angle Z, \angle X, \angle Y$

5.



$\angle J, \angle L, \angle K$

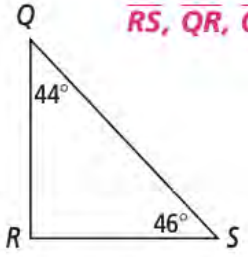
6.



$\angle Q, \angle S, \angle R$

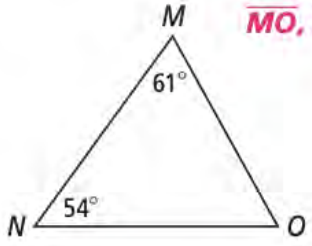
For Exercises 7–10, list the sides of each triangle in order from shortest to longest.

7.



$\overline{RS}, \overline{QR}, \overline{QS}$

8.



$\overline{MO}, \overline{NO}, \overline{MN}$

9. $\triangle ABC$, with $m\angle A = 99$, $m\angle B = 44$, and $m\angle C = 37$ $\overline{BA}, \overline{AC}, \overline{CB}$

10. $\triangle ABC$, with $m\angle A = 122$, $m\angle B = 22$, and $m\angle C = 36$ $\overline{AC}, \overline{BA}, \overline{CB}$

For Exercises 11 and 12, list the angles of each triangle in order from smallest to largest.

11. $\triangle ABC$, where $AB = 17$, $AC = 13$, and $BC = 29$ $\angle B, \angle C, \angle A$

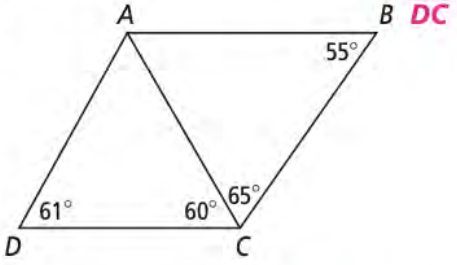
12. $\triangle MNO$, where $MN = 4$, $NO = 12$, and $MO = 10$ $\angle O, \angle N, \angle M$

5-6 Practice (continued)

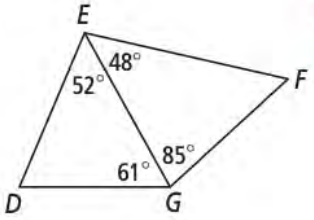
Form G

Inequalities in One Triangle

Determine which side is shortest in the diagram.

13. 

$\overline{DC}$

14. 

$\overline{DG}$

Can a triangle have sides with the given lengths? Explain.

15. 8 cm, 7 cm, 9 cm Yes; the sum of the lengths of any two sides is always greater than the length of the third side.

16. 7 ft, 13 ft, 6 ft No; the sum of the lengths of the two shorter sides is equal to the length of the third side; $6 + 7 = 13$

17. 20 in., 18 in., 16 in. Yes; the sum of the lengths of any two sides is always greater than the length of the third side.

18. 3 m, 11 m, 7 m No; the sum of the lengths of the two shorter sides is less than the length of the third side; $7 + 3 < 11$

Algebra The lengths of two sides of a triangle are given. Describe the possible lengths for the third side.

19. 5, 11 any length that is greater than 6 and less than 16

20. 12, 12 any length that is greater than 0 and less than 24

21. 25, 10 any length that is greater than 15 and less than 35

22. 6, 8 any length that is greater than 2 and less than 14

23. Algebra List the sides in order from shortest to longest in $\triangle PQR$, with $m\angle P = 45$, $m\angle Q = 10x + 30$, and $m\angle R = 5x$. $\overline{PQ}$, $\overline{QR}$, $\overline{PR}$

24. Algebra List the sides in order from shortest to longest in $\triangle ABC$, with $m\angle A = 80$, $m\angle B = 3x + 5$, and $m\angle C = 5x - 1$. $\overline{AC}$, $\overline{BA}$, $\overline{BC}$

25. Error Analysis A student draws a triangle with a perimeter 36 cm. The student says that the longest side measures 18 cm. How do you know that the student is incorrect? Explain.
The Triangle Inequality Theorem says that the sum of any two sides must be greater than the third side. If one side measures 18 cm and the perimeter is 36 cm, then the other two sides must sum to 18 cm, but they must add up to greater than 18.

5-6 Practice

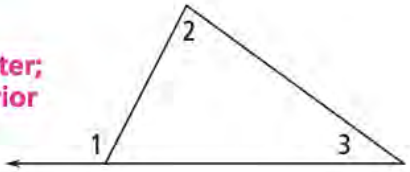
Form K

Inequalities in One Triangle

1. Explain the relationship of $m\angle 1$, $m\angle 2$, and $m\angle 3$.

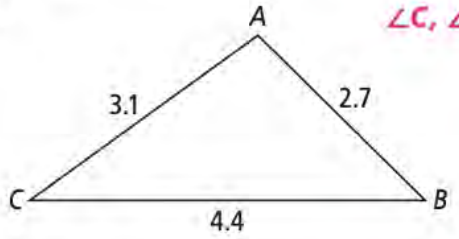
The measure of an exterior angle of a triangle is ___?___ than the measure of each of its remote ___?___ angles. **greater; interior**

$\angle 1$ is an ___?___ angle of the triangle, so $m\angle 1 >$ ___?___ and $m\angle 1 >$ ___?___. **exterior; $m\angle 2$; $m\angle 3$**



For Exercises 2–5, list the angles of each triangle in order from smallest to largest.

2.

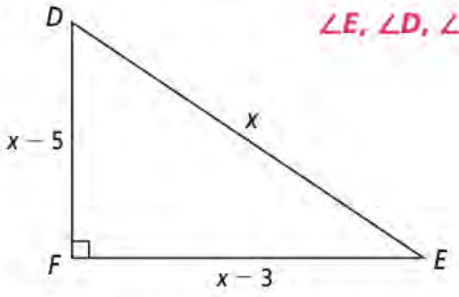


$\angle C$, $\angle B$, $\angle A$

To start, order the side lengths from least to greatest.

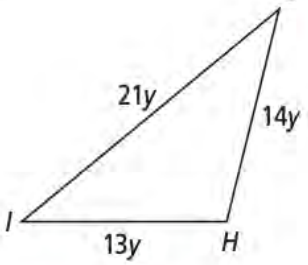
2.7 < **3.1** < **4.4**

3.



$\angle E$, $\angle D$, $\angle F$

4.

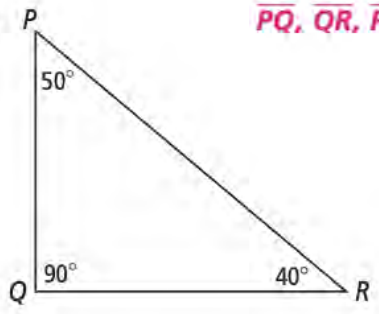


$\angle G$, $\angle I$, $\angle H$

5. $\triangle XYZ$, where $XY = 25$, $YZ = 11$, and $XZ = 15$ **$\angle X$, $\angle Y$, $\angle Z$**

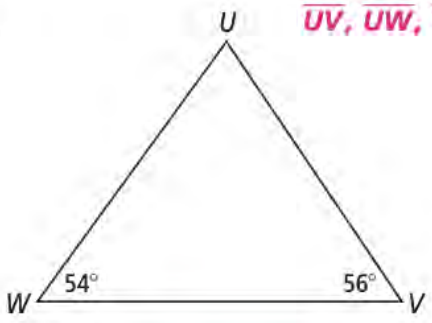
For Exercises 6–8, list the sides of each triangle in order from shortest to longest.

6.



$\overline{PQ}$, $\overline{QR}$, $\overline{PR}$

7.



$\overline{UV}$, $\overline{UW}$, $\overline{WV}$

8. $\triangle MNO$, where $m\angle M = 56$, $m\angle N = 108$, and $m\angle O = 16$ **$\overline{MN}$, $\overline{NO}$, $\overline{MO}$**

9. Algebra List the sides in order from shortest to longest in $\triangle XYZ$, with $m\angle X = 50$, $m\angle Y = 5x + 10$, and $m\angle Z = 5x$. **$\overline{YZ}$, $\overline{XY}$, $\overline{XZ}$**

5-6 Practice (continued)

Form K

Inequalities in One Triangle

Can a triangle have sides with the given lengths? Explain.

10. 10 in., 13 in., 18 in.

To start, choose two sides and see if their sum exceeds the third side.

$10 + 13 [\overset{?}{>}] 18$ (yes) / no (Circle the correct answer.)

Check the other two sums. **Yes; the sum of the lengths of the two shorter sides is always greater than the length of the third side.**

11. 6 m, 5 m, 12 m **No; the sum of the lengths of the two shorter sides is less than the length of the third side; $6 + 5 < 12$.**

12. 11 ft, 8 ft, 18 ft **Yes; the sum of the lengths of the two shorter sides is always greater than the length of the third side.**

Algebra The lengths of two sides of a triangle are given. Find the range of possible lengths for the third side.

13. 4, 8 **any length that is greater than 4 and less than 12**

To start, write the inequalities relating the known side lengths and the unknown side length.

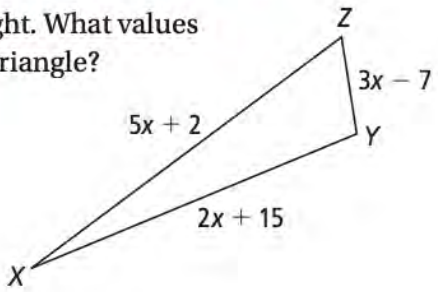
$x + 4 > 8$ $\qquad x + 8 > \boxed{4}$ $\qquad 8 + 4 > \boxed{x}$

14. 13, 8 **any length that is greater than 5 and less than 21**

15. 10, 15 **any length that is greater than 5 and less than 25**

16. Error Analysis A student draws a triangle with a perimeter of 12 in. The student says that the longest side measures 7 in. How do you know that the student is incorrect? Explain. **If one side measures 7 in., the other two side measures must sum to 5 in., but the Triangle Inequality Theorem says that the sum of the measures must be greater than 7 in.**

17. Algebra $\triangle XYZ$ has the side lengths shown at the right. What values of x result in side lengths that could be the sides of a triangle? (*Hint:* Write and solve three inequalities.) $x > \frac{10}{3}$



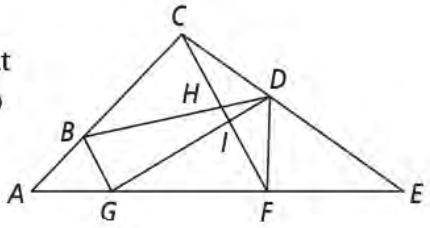
5-6 Enrichment

Inequalities in One Triangle

The First of the Greek Geometers

Euclid organized the known geometry of his day into one of history's all-time best sellers, *The Elements.* However, Euclid was not the father of Greek geometry. That honor belongs to a philosopher known as one of the Seven Wise Men of ancient Greece, who first introduced geometry to Greece. He founded the geometry of lines and was probably the first great mathematician in history.

To discover the identity of this great mathematician and philosopher, consider this diagram in which every angle that appears to be acute is acute, and every angle that appears to be obtuse is obtuse. Answer each question individually, and without the use of a ruler or a protractor.



Each statement below is true (T), false (F), or undecidable (U). The answers to Exercises 1–3 will supply the hint to decode the first letter of the name, the answers to Exercises 4–6 to the second letter, and so on.

A – TTT	B – TTF	C – TTU	D – TFT	E – TFF	F – TFU	G – TUT
H – TUF	I – TUU	J – FTT	K – FTF	L – FTU	M – FFT	N – FFF
O – FFU	P – FUT	Q – FUF	R – FUU	S – UTT	T – UTF	U – UTU
V – UFT	W – UFF	X – UFU	Y – UUT	Z – UUF		

Relation	T, U, or F	Relation	T, U, or F
1. $BG > DF$	? U	10. $m\angle AGB > m\angle AGD$	? F
2. $CF > DF$	? T	11. $m\angle CBG > m\angle BAG$	? T
3. $m\angle CEF > m\angle CFE$	? F	12. $AG > CH$	? U
4. $m\angle BHC > m\angle CDH$	? T	13. $CF > HI$	? T
5. $DG > DE$	? U	14. $m\angle CFD > m\angle AFD$	? F
6. $m\angle DGF > m\angle DFE$	? F	15. $BH > BD$	? F
7. $DG > IG$	? T	16. $BH > EF$	? U
8. $m\angle AFD > m\angle AEC$	? T	17. $CE > CF$	? T
9. $m\angle CDF > m\angle GBD$	? T	18. $m\angle ACE > m\angle ACF$	? T

The name of the mathematician is T H A L E S (thā' lēz).

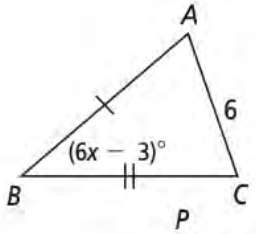
19. List at least five reasons that you used to determine whether the statements above were true, false, or undecidable.

Answers may vary. Sample: Theorem 5-10, Theorem 5-11, Angle Addition Postulate, Segment Addition Postulate, Comparison Property of Inequality, Transitive Property of Inequality, and an exterior angle of a triangle is greater than either remote interior angle.

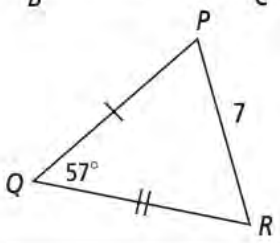
5-7 Additional Vocabulary Support

Inequalities in Two Triangles

You want to use the Converse of the Hinge Theorem to find the possible values for x in the triangle at the right.



You wrote these steps to solve the problem on the note cards, but they got mixed up.



Solve the first inequality:

$6x < 60$, so

$x < 10$.

Write two inequalities:

$6x - 3 < 57$ and

$6x - 3 > 0$.

Solve the second inequality:

$6x > 3$, so

$x > 0.5$.

Compare measures of the included angles.

$PR > AC$, so

$m\angle Q > m\angle B$.

Think: The triangles ABC and PQR have two pairs of congruent sides. Identify the included angles: $\angle B$ and $\angle Q$.

Write an inequality for the possible values of x:

$0.5 < x < 10$.

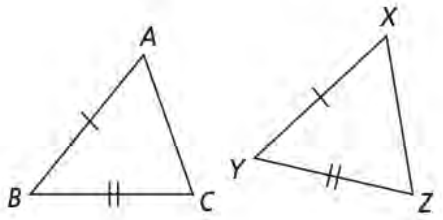
Use the note cards to write the steps in order.

1. First, think: The triangles *ABC* and *PQR* have two pairs of congruent sides. Identify the included angles: $\angle B$ and $\angle Q$.
2. Second, compare measures of the included angles. $PR > AC$, so $m\angle Q > m\angle B$.
3. Third, write two inequalities: $6x - 3 < 57$ and $6x - 3 > 0$.
4. Fourth, solve the first inequality: $6x < 60$, so $x < 10$.
5. Next, solve the second inequality: $6x > 3$, so $x > 0.5$.
6. Then, write an inequality for the possible values of x: $0.5 < x < 10$.

5-7 Reteaching

Inequalities in Two Triangles

Consider $\triangle ABC$ and $\triangle XYZ$. If $\overline{AB} \cong \overline{XY}$, $\overline{BC} \cong \overline{YZ}$, and $m\angle Y > m\angle B$, then $XZ > AC$. This is the *Hinge Theorem* (SAS Inequality Theorem).



Problem

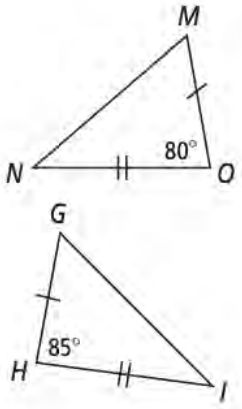
Which length is greater, GI or MN?

Identify congruent sides: $\overline{MO} \cong \overline{GH}$ and $\overline{NO} \cong \overline{HI}$.

Compare included angles: $m\angle H > m\angle O$.

By the Hinge Theorem, the side opposite the larger included angle is longer.

So, $GI > MN$.



Problem

At which time is the distance between the tip of a clock's hour hand and the tip of its minute hand greater, 3:00 or 3:10?

Think of the hour hand and the minute hand as two sides of a triangle whose lengths never change, and the distance between the tips of the hands as the third side. 3:00 and 3:10 can then be represented as triangles with two pairs of congruent sides. The distance between the tips of the hands is the side of the triangle opposite the included angle.

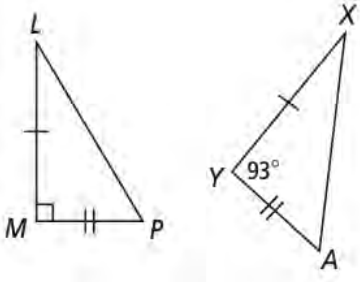
At 3:00, the measure of the angle formed by the hour hand and minute hand is 90°. At 3:10, the measure of the angle is less than 90°.

So, the distance between the tip of the hour hand and the tip of the minute hand is greater at 3:00.

Exercises

1. What is the inequality relationship between LP and XA in the figure at the right? ***XA > LP***

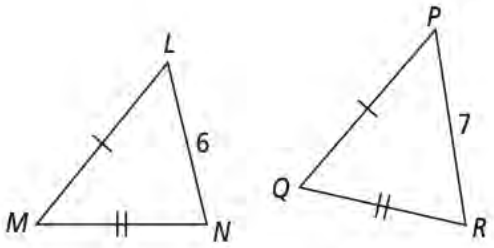
2. At which time is the distance between the tip of a clock's hour hand and the tip of its minute hand greater, 5:00 or 5:15? **5:00**



5-7 Reteaching (continued)

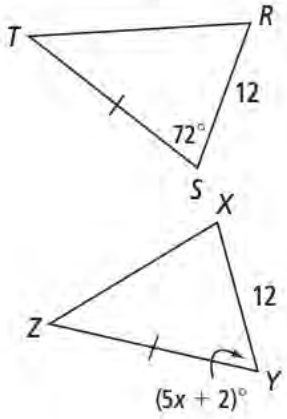
Inequalities in Two Triangles

Consider $\triangle LMN$ and $\triangle PQR$. If $\overline{LM} \cong \overline{PQ}$, $\overline{MN} \cong \overline{QR}$, and $PR > LN$, then $m\angle Q > m\angle M$. This is the *Converse of the Hinge Theorem* (SSS Inequality Theorem).



Problem

$TR > ZX$. What is the range of possible values for x?



The triangles have two pairs of congruent sides, because $RS = XY$ and $TS = ZY$. So, by the Converse of the Hinge Theorem, $m\angle S > m\angle Y$.

Write an inequality:

$72 > 5x + 2$	Converse of the Hinge Theorem
$70 > 5x$	Subtract 2 from each side.
$14 > x$	Divide each side by 5.

Write another inequality:

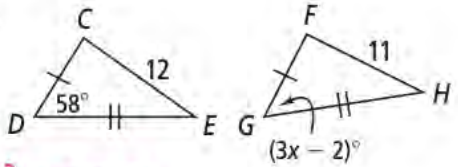
$m\angle Y > 0$	The measure of an angle of a triangle is greater than 0.
$5x + 2 > 0$	Substitute.
$5x > -2$	Subtract 2 from each side.
$x > -\frac{2}{5}$	Divide each side by 5.

So, $-\frac{2}{5} < x < 14$.

Exercises

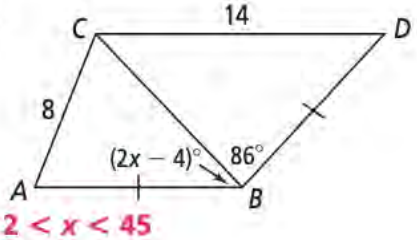
Find the range of possible values for each variable.

3.



$\frac{2}{3} < x < 20$

4.



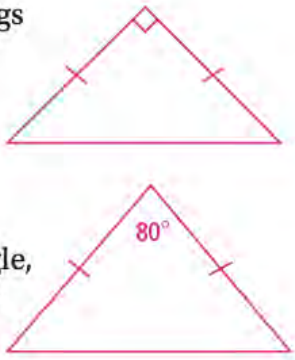
$2 < x < 45$

5. Reasoning An equilateral triangle has sides of length 5, and an isosceles triangle has side lengths of 5, 5, and 4. Write an inequality for x, the measure of the vertex angle of the isosceles triangle. $60 > x > 0$

5-7 Think About a Plan

Inequalities in Two Triangles

Reasoning The legs of a right isosceles triangle are congruent to the legs of an isosceles triangle with an 80° vertex angle. Which triangle has a greater perimeter? How do you know?



1. How can you use a sketch to help visualize the problem?
   Draw a sketch.

2. The triangles have two pairs of congruent sides. For the right triangle, what is the measure of the included angle? How do you know this?
   90; it is a right angle.

3. For the second triangle, what is the measure of the included angle? How do you know this?
   80; it is given that the vertex angle, the angle between the legs, or congruent sides, is 80°.

4. How could you find the perimeter of each triangle?
   Add the lengths of the legs and the base.

5. How does the sum of the lengths of the legs in the right triangle compare to the sum of the lengths of the legs in the other triangle?
   They are the same.

6. Write formulas for the perimeters of each triangle. Use the variable ℓ for leg length, b_1 for base length of the right triangle, and b_2 for base length of the second triangle.
   P(right triangle) $= \ell + \ell + b_1$; P(second triangle) $= \ell + \ell + b_2$

7. What values do you need to compare to find the triangle with the greater perimeter?
   base lengths, or b_1 and b_2

8. How can you use the Hinge Theorem to find which base length is longer?
   The longer base length will be opposite the larger included angle.

9. Which base length is longer? b_1, or the base length of the right triangle

10. Which triangle has the greater perimeter? the right isosceles triangle

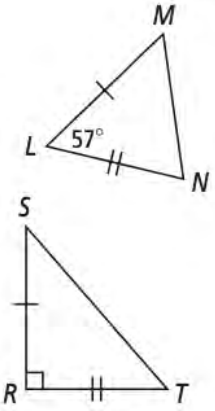
5-7 Practice

Form G

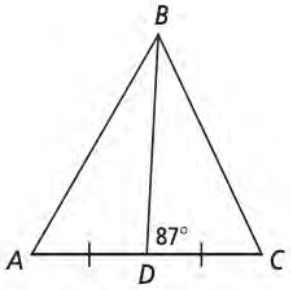
Inequalities in Two Triangles

Write an inequality relating the given side lengths. If there is not enough information to reach a conclusion, write *no conclusion.*

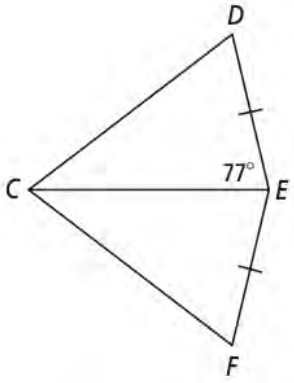
1. ST and MN **ST > MN**



2. BA and BC **BA > BC**



3. CD and CF **no conclusion**



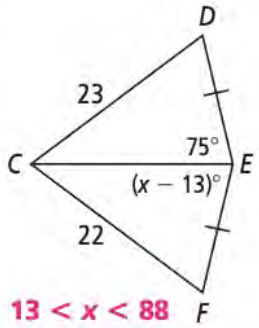
4. A crocodile opens his jaws at a 30° angle. He closes his jaws, then opens them again at a 36° angle. In which case is the distance between the tip of his upper jaw and the tip of his lower jaw greater? Explain.

The distance is greater when the jaw is opened 36°. The jawbones are congruent pairs of sides of a triangle. The jaw joint is the included angle between the sides. The distance between the tips of the upper jaw and lower jaw is the length of a third side of the triangle. This is greatest when the angle opposite it is greatest.

5. At which time is the distance between the tip of the hour hand and the tip of the minute hand greater, 2:20 or 2:25? **2:25**

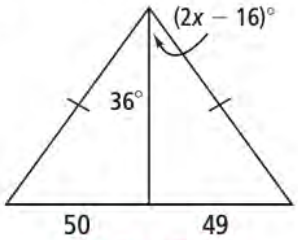
Find the range of possible values for each variable.

6.



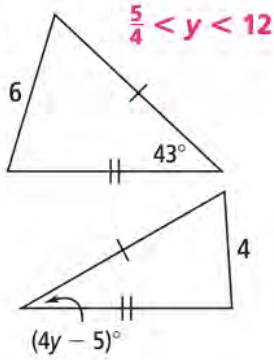
13 < x < 88

7.

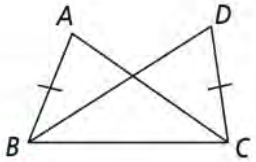


8 < x < 26

8. $\frac{5}{4} < y < 12$



9. In the triangles at the right, $AB = DC$ and $m\angle ABC < m\angle DCB$. Explain why $AC < BD$.

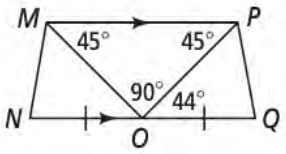


It is given that *AB = DC*. *BC = BC* by the Refl. Prop. of =. Therefore, these two triangles have two pairs of congruent sides. The included angles in *△ABC* and *△BDC* are *∠ABC* and *∠DCB*. It is given that *m∠ABC < m∠DCB*, so by the Hinge Theorem, *AC < BD*.

5-7 Practice (continued) Form G

Inequalities in Two Triangles

Copy and complete with > or <. Explain your reasoning.



10. $m\angle POQ$? $m\angle MON$

<; the measure of ∠MON is 46, because it forms a straight angle with ∠MOP and ∠POQ.

11. MN ? PQ

>; because △MPO is an isosceles triangle, MO = PO. △MNO and △POQ have two pairs of congruent sides. In △MON the included angle is ∠MON and in △POQ the included angle is ∠POQ. m∠MON > m∠POQ, so by the Hinge Theorem, MN > PQ.

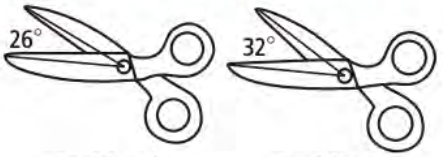
12. MP ? OP

>; the longest leg of a △ is opposite the angle with the greatest measure.

13. Jogger A and Jogger B start at the same point. Jogger A travels 0.9 mi due east, then turns 120° clockwise, then travels another 3 mi. Jogger B travels 0.9 mi due west, then turns 115° counterclockwise, then travels another 3 mi. Do the joggers end in the same place? Explain. **No; the direct distance between the start and end points is the length of the third leg of each jogger's △. For Joggers A and B, the △s have two pairs of congruent legs. The measure of the included angle is less for Jogger A than for Jogger B, so the direct distance between the start and end points is less for Jogger A than for Jogger B.**

14. In the diagram at the right, in which position are the tips of the scissors farther apart?

Position B



Position A Position B

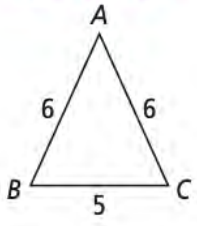
15. The legs of an isosceles triangle with a 65° vertex angle are congruent with the sides of an equilateral triangle. Which triangle has a greater perimeter? How do you know?

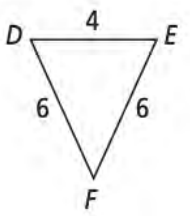
The isosceles △; the △s of an equilateral △ all measure 60. Because 65 > 60, the third side of the isosceles △ is longer than the third side of the equilateral △. So, the isosceles △ will have a greater perimeter.

Write an inequality relating the given angle measures. If there is not enough information to reach a conclusion, write *no conclusion*.

16. $m\angle A$ and $m\angle F$

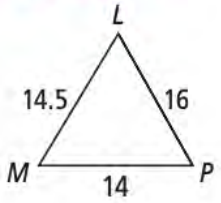
m∠A > m∠F

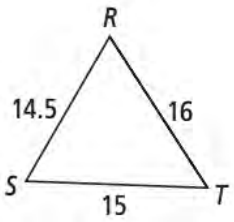




17. $m\angle L$ and $m\angle R$

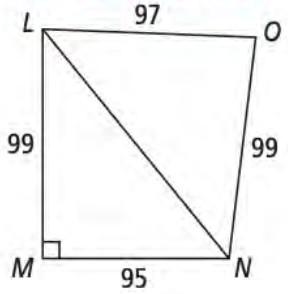
m∠L < m∠R





18. $m\angle MLN$ and $m\angle ONL$

m∠MLN < m∠ONL



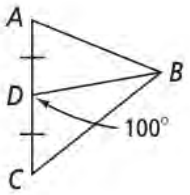
5-7 Practice

Form K

Inequalities in Two Triangles

Write an inequality relating the given side lengths. If there is not enough information to reach a conclusion, write *no conclusion*.

1. AB and CB **AB < CB**



To start, determine whether the triangles have two pairs of congruent sides.

$\overline{AD} \cong \overline{CD}$

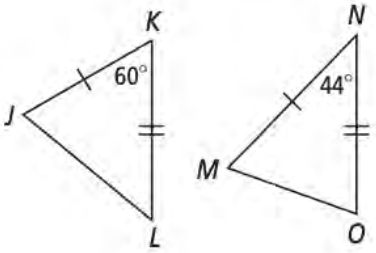
$\overline{DB} \cong$? **DB**

Then compare the hinge angles.

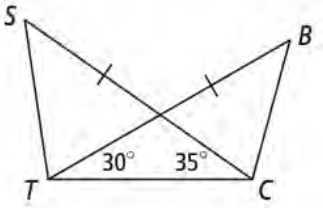
$m\angle CDB =$ **100**

$m\angle$ **ADB** $=$ **80**

2. JL and MO **JL > MO**

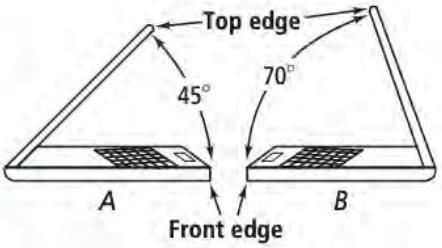


3. ST and BT **no conclusion**



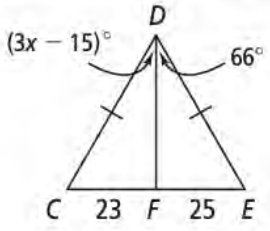
4. Two identical laptops are shown at the right. In which laptop is the distance from the top edge of the screen to the front edge of the keyboard greater? Explain.

Laptop B; the lengths of the laptops' keyboards and screens are the same. Laptop B is open wider, so the hinge angle is greater than it is for Laptop A.



Algebra Find the range of possible values for each variable.

5.



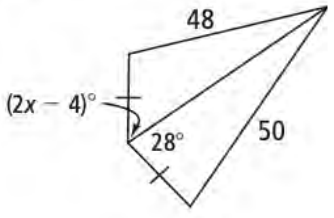
$m\angle CDF < m\angle EDF$

$x <$ **27**

$m\angle CDE > 0$

$x >$ **5**

6.



2 $< x <$ **16**

5-7 Enrichment

Inequalities in Two Triangles

Comparing Distances

Triangle inequalities can be used to solve real-world problems like the following:

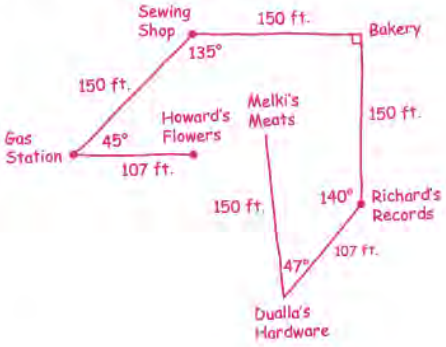
Samantha works at a bakery, and delivers pastries to local businesses.

On Tuesdays and Thursdays, she walks due west out of the bakery and travels 150 ft to the Sewing Shop. From there, she turns 45° toward the south, and travels another 150 ft to the gas station. From the gas station she travels 107 ft due east to Howard's Flowers. Howard's Flowers is due south of the Sewing Shop.

On Mondays and Wednesdays, Samantha travels 150 ft due south of the bakery to Richard's Records. From there, she makes a 40° turn toward the west and travels 107 ft to Dualla's Hardware.

On Fridays, Samantha makes the Monday-Wednesday route, but also goes to Melki's Meats. From the hardware store, she turns 133° to her right and travels 150 ft northeast to Melki's.

1. Make a drawing of Samantha's delivery routes. Label all angles and all distances between buildings.



2. Use your drawing to answer the following questions. Assume a straight path between each set of points.
   a. Which is greater, the distance between the Sewing Shop and Howard's Flowers, or the distance between Richard's Records and Melki's Meats?
   the distance between Richard's Records and Melki's Meats

   b. Which is greater, the distance between the bakery and Howard's Flowers, or the distance between Howard's Flowers and the gas station?
   the distance between the bakery and Howard's Flowers

3. One day Samantha has to make a delivery from the bakery to Richard's Records, then to Melki's Meats. What is the minimum distance she must walk to get to Melki's via Richard's? Explain.
   At least 257 ft; the distance between Richard's and Melki's is more than the distance between the Sewing Shop and Howard's Flowers, which is 107 ft. The distance between the bakery and Richard's is 150 ft. So, she must walk *at least* 150 + 107 = 257 ft.